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N. LISKOV

Analytical Techniques for Linear Time-Varying Systems

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ANALYTICAL TECHNIQUES FOR LINEAR TIME-VARYING SYSTEMS

N. Liskov

SYSTEM THEORY RESEARCH

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ABSTRACT

The definition of several new system functions leads to a more complete characterization of time-varying linear systems. A family of twelve system functions, including the impulse-response K system functions, are used to describe time-varying linear systems. The relationships among the various system functions are clearly illustrated. The K system functions are shown to be convenient for the analysis of cascaded systems. The time-frequency duality concept is discussed with respect to the system functions, and the introduction of physical variables extends the duality concept so that knowing one relationship is equivalent to knowing four relationships.

The expansion of system functions in terms of a complete set of functions in two dimensional space or in a sampling series for an appropriately band-limited and/or time-limited system leads to a matrix characterization of time-varying linear systems. Schemes for evaluating the coefficients of the expansions are described. The causality condition requires that the matrix of $k(t,\tau)$ be lower triangular. Marcovitz's findings, using the matrix of $k(t,\tau)$, on the conditions for the existence of quasi-inverses and inverses to time-varying systems are extended to the recoverability of a signal of finite duration. The matrix formulation is applied to the problem of finding eigenfunctions and eigenvalues for time-varying systems. When one knows the eigenvectors of a time-varying system, the input-output relationship is greatly simplified.

When the $h(t, v)$ function is expanded in a series in terms of a complete set of realizable networks $\{\psi_i(v)\}$, it can be realized by a parallel combination of h-separable networks in which a typical branch consists of the network $\psi_i(v)$ followed by the multiplier $a_i(t)$. Because of the difficulty in synthesizing an arbitrary multiplier, one considers the double expansion of $h(t, v)$ in terms of the complete set $\{\psi_i(t)\psi_j(v)\}$ where one has some choice in the set of multipliers $\{\phi_i(t)\}$. In particular, if the multipliers are sisoids the multiplier becomes a standard modulator network. A unique realization, based on the double expansion, is presented where the networks $\{\psi_i(v)\}$ are connected to the multipliers $\{\phi_i(t)\}$ by a resistive coefficient matrix. A realization scheme using the link structure and based on the sampling theorem is also presented.

The characterization and analysis methods are shown to be applicable to characterizing a satellite communication system and to correcting for delay and Doppler in an air-to-ground communications system and in a side-looking radar surveillance system. The matrix methods are applied to a feedback control system for a time-varying plant and to finding the system matrix of a whitening filter whose output is white stationary noise when the input is nonstationary noise with a given correlation $R_n(t,\tau)$. The matched filter for a channel with nonstationary noise is shown to consist of the whitening filter followed in cascade by a time-varying matched filter that is matched to the signal component of the output of the whitening filter. When the noise is slowly varying, the optimum transmitted signal is found.

I. INTRODUCTION

Linear time-varying systems have received considerable attention in recent years. Besides the intellectual challenge of the extension of system theory that linear time-varying systems present to the theorists, there is an increasing practical need for techniques of analyzing and synthesizing these systems. Description and analysis of physical systems by time-varying models, analysis of existing time-varying systems, more effective use of physical devices exhibiting time-varying characteristics, and adaptive feedback control of time-varying systems are some examples of practical as well as theoretical interest.

Little work in time-varying systems had been done from the system theory point of view before 1950 when Zadeh[1] introduced the H system function. Since then, however, a number of articles on time-varying system theory have appeared in the liaterature. The G system function that is the dual of the H-system function was introduced by Gersho,[2] and more system functions have been introduced by others.[3,4] In this connection, time-frequency duality seems to play an important role.[5] A clear description of all the various system functions and the transform, functional, and dual relationships among them is now possible.

A. THE RELATIONSHIPS AMONG SYSTEM FUNCTIONS OF TIME-VARYING SYSTEMS

In the study of time-varying systems, various system functions

a rise that are useful in characterizing the system and in determining the output from a given input. For a given system, one system function may be easier to find or work with than another system function. All the system functions are now defined, and the attempt is made to show clearly the relationships which exist among all the system functions. This information will be useful for finding dual results, and for finding one system function when another system function is given.

In Figure 1.1, the relationships among the system functions are shown in diagram form. Some of these functions are familiar, but others that are new are explained here.

The system functions in the time-time domain give the system output in time, to an impulse in time applied at the input. These functions are:

$k(t,\tau)$ = output at time t to an input applied at time τ,

$g(v,\tau)$ = output v seconds after an input applied at time τ,

$h(t,r)$ = output at time t to an input applied r seconds ago.

The time-shift or "age" variables are

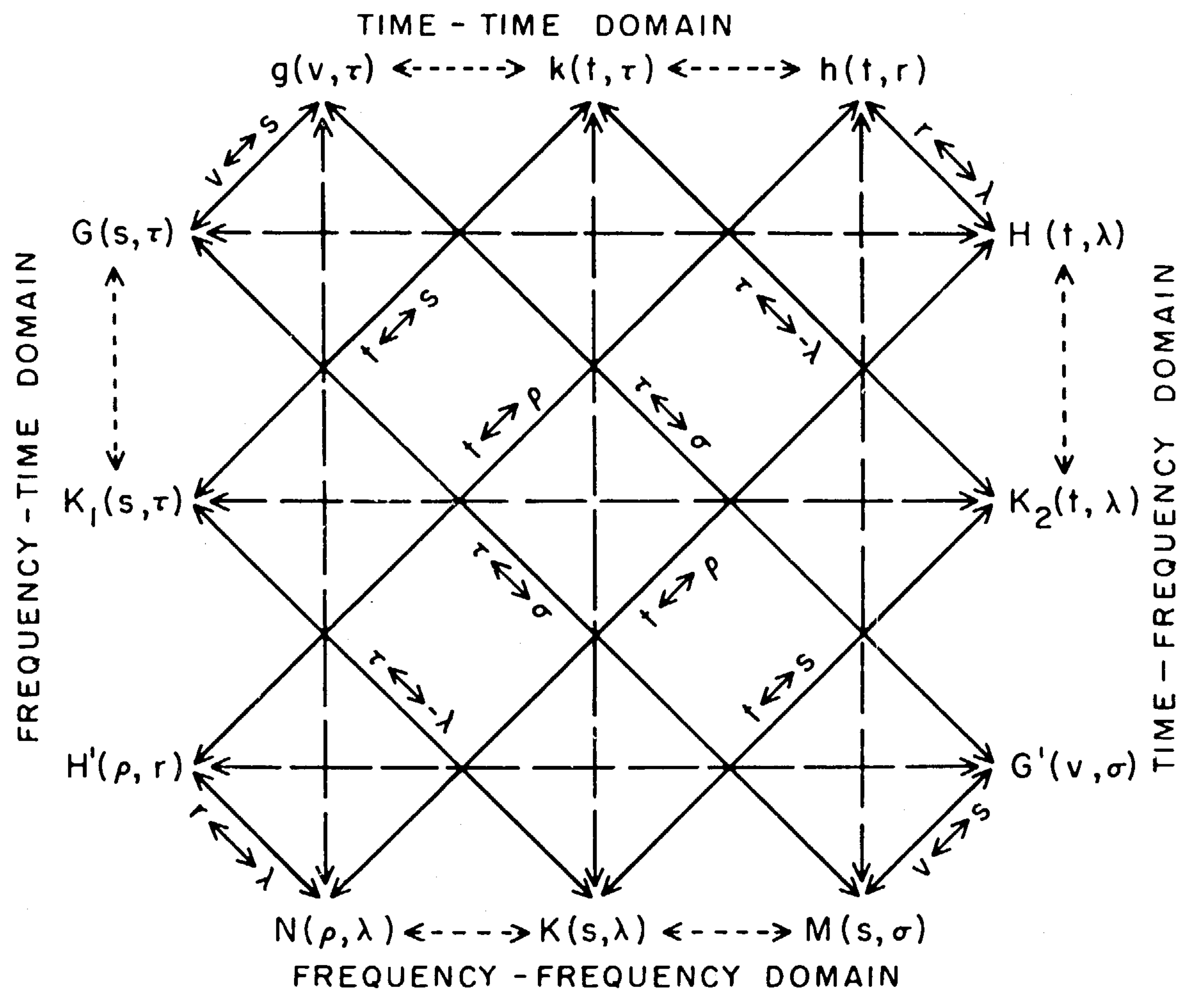
$$v = r = t - \tau \quad . \tag{1.1}$$

The impulse-response functions in time are therefore related by the changes of variables,

$$g(v,\tau) = k(\tau + v, \tau) \quad , \tag{1.2}$$

and

$$h(t,r) = k(t, t-r) \quad . \tag{1.3}$$



——— LAPLACE TRANSFORMS
— — DUALS
- - - - - - FUNCTIONAL RELATIONSHIPS

FIGURE 1.1. Relationships among System Functions.
Dual variables: $\begin{matrix} v & \tau & t & r \\ \rho & \lambda & s & \sigma \end{matrix}$

Note that the function $h(t, r)$ has its variables interchanged from previous definitions, later leading to the fact that corresponding variables of dual functions are duals. This was not true before, unless one relabeled the variables of either the function or its dual.

The system functions in the frequency-frequency domain give the system output in frequency, to an impulse in frequency applied at the input. These functions are:

$K(s,\lambda)$ = output at frequency s to an input at frequency λ ,

$N(\rho, \lambda)$ = output at ρ cycles higher than the input frequency λ ,

$M(s,\sigma)$ = output at frequency s to an input at σ cycles lower.

The frequency-shift variables are

$$\rho = \sigma = s - \lambda \quad . \tag{1.4}$$

The impulse-response functions in frequency are therefore related by the changes of variables,

$$N(\rho, \lambda) = K(\rho + \lambda, \lambda) \quad , \tag{1.5}$$

and

$$M(s,\sigma) = K(s, s - \sigma) \quad . \tag{1.6}$$

The frequency-impulse-response functions K, N and M are, respectively, the double Laplace transforms of the time-impulse-response functions k, h and g . The functions $k(t, \tau)$ and $K(s, \lambda)$, $h(t, r)$ and $M(s,\sigma)$, and $g(v, \tau)$ and $N(\rho, \lambda)$ are duals, and their corresponding variables are dual variables.

The functions in the time-frequency and frequency-time domains are of a mixed nature and must be discussed separately. The familiar system functions $G(s, \tau)$ and $H(t, \lambda)$ are defined by a single transform of $g(v, \tau)$ and $h(t, r)$, respectively. These system functions are convenient because they look like system functions of time-invariant systems. This is evident from the alternate definitions of the $G(s, \tau)$ and $H(t, \lambda)$ system functions

$$G(s,\tau) = \left.\frac{Y(s)}{X(s)}\right|_{X(s) = e^{-s\tau}} , \tag{1.7}$$

and

$$H(t, \lambda) = \left.\frac{y(t)}{x(t)}\right|_{x(t) = e^{\lambda t}} . \tag{1.8}$$

The functions $G(s, \tau)$ and $H(t, \lambda)$ are duals.

The functions $K_1(s, \tau)$ and $K_2(t, \lambda)$ are defined as single transforms of $k(t, \tau)$ by

$$K_1(s, \tau) = \int_{-\infty}^{\infty} k(t, \tau)\, e^{-st}\, dt , \tag{1.9}$$

and

$$K_2(t, \lambda) = \int_{-\infty}^{\infty} k(t, \tau)\, e^{\lambda \tau}\, d\tau . \tag{1.10}$$

It can be shown easily that

$$K_1(s, \tau) = G(s, \tau)\, e^{-s\tau} = Y(s)\Big|_{\substack{x(s) = e^{-s\tau} \\ [x(t) = \delta(t - \tau)]}} , \tag{1.11}$$

and

$$K_2(t,\lambda) = H(t,\lambda)\, e^{\lambda t} = y(t) \Bigg|_{\substack{x(t) = e^{\lambda t} \\ [X(s) = 2\pi\delta(s-\lambda)]}} \quad . \tag{1.12}$$

Now a physical interpretation can be given to $K_1(s,\tau)$ and $K_2(t,\lambda)$. The function $K_1(s,\tau)$ is the output in frequency to an impulse in time which is applied at time $t=\tau$, and the function $K_2(t,\lambda)$ is the output in time to an impulse in frequency applied at frequency $s=\lambda$. The system output is given by

$$Y(s) = \int_{-\infty}^{\infty} K_1(s,\tau)\, x(\tau)\, d\tau \quad , \tag{1.13}$$

and

$$y(t) = \int_{-\infty}^{\infty} K_2(t,\lambda)\, X(\lambda)\, d\lambda \quad . \tag{1.14}$$

The mixed-impulse-response functions $K_1(s,\tau)$ and $K_2(t,\lambda)$ are duals. One convenience of the interpretation of the K functions as impulse responses is that one has to remember only one input-output equation, instead of four:

$$\begin{aligned} y(\text{output variable}) = \int & \gamma(\text{output variable, input variable}) \\ & x(\text{input variable})\, d(\text{input variable}) \quad , \end{aligned} \tag{1.15}$$

where γ is the K-system function which is appropriate for its variables.

The complimentary system functions $H'(\rho, t)$ and $G'(v, \sigma)$ are defined according to Figure 1.1 as

$$H'(\rho, r) = \int_{-\infty}^{\infty} h(t, r)\, e^{-\rho t}\, dt \quad , \tag{1.16}$$

and

$$G'(v, \sigma) = \int_{-\infty}^{\infty} g(v, \tau)\, e^{-\sigma \tau}\, d\tau \quad . \tag{1.17}$$

No physical significance for the complimentary system functions has been found yet, but they are still effective in characterizing the system because they are uniquely related to the other system functions. The complimentary system functions $H'(\rho, r)$ and $G'(v, \sigma)$ are duals, and the following relationship holds between them:

$$H'(\rho, r) = G'(r, \rho)\, e^{-r\rho} \quad . \tag{1.18}$$

B. DRIVING-POINT IMMITTANCE FUNCTIONS OF LINEAR TIME-VARYING SYSTEMS

The G, H and K system functions give relationships between input and output variables of a system, without regard to the physical quantities that the variables represent. The input and output variables in the driving-point case represent voltage and current, and the G, H and K system functions take on an impedance or admittance character. The driving-point immittance functions are defined by

$$G_Y(s,\tau) = \left.\frac{I(s)}{E(s)}\right|_{E(s) = e^{-s\tau}} \qquad H_Y(t,\lambda) = \left.\frac{i(t)}{e(t)}\right|_{e(t) = e^{\lambda t}} , \tag{1.19}$$

$$G_Z(s,\tau) = \left.\frac{E(s)}{I(s)}\right|_{I(s) = e^{-s\tau}} \qquad H_Z(t,\lambda) = \left.\frac{e(t)}{i(t)}\right|_{i(t) = e^{\lambda t}} , \tag{1.20}$$

$$K_{1Y}(s,\tau) = I(s)\Big|_{e(t) = \delta(t-\tau)} \qquad K_{2Y}(t,\lambda) = i(t)\Big|_{E(s) = 2\pi\delta(s-\lambda)} , \tag{1.21}$$

$$K_{1Z}(s,\tau) = E(s)\Big|_{i(t) = \delta(t-\tau)} \qquad K_{2Z}(t,\lambda) = e(t)\Big|_{I(s) = 2\pi\delta(s-\lambda)} . \tag{1.22}$$

From the definition of the K function and its physical significance, it is clear that the time-to-time and frequency-to-frequency impulse-response functions are

$$K_Y(s,\lambda) = I(s)\Big|_{E(s) = 2\pi\delta(s-\lambda)} \qquad k_Y(t,\tau) = i(t)\Big|_{e(t) = \delta(t-\tau)} , \tag{1.23}$$

$$K_Z(s,\lambda) = E(s)\Big|_{I(s) = 2\pi\delta(s-\lambda)} \qquad k_Z(t,\tau) = e(t)\Big|_{i(t) = \delta(t-\tau)} . \tag{1.24}$$

Relationships between the driving-point variables are found

from the above definitions and from previously obtained equations relating input and output variables. They are

$$i(t) = \int H_Y(t,\lambda)\, E(\lambda)\, e^{\lambda t}\, d\lambda = \int K_{2Y}(t,\lambda)\, E(\lambda)\, d\lambda = \int k_Y(t,\tau)\, e(\tau)\, d\tau \quad , \tag{1.25}$$

$$e(t) = \int H_Z(t,\lambda)\, I(\lambda)\, e^{\lambda t}\, d\lambda = \int K_{2Z}(t,\lambda)\, I(\lambda)\, d\lambda = \int k_Z(t,\tau)\, i(\tau)\, d\tau \quad , \tag{1.26}$$

$$I(s) = \int G_Y(s,\tau)\, e(\tau)\, e^{-s\tau}\, d\tau = \int K_{1Y}(s,\tau)\, e(\tau)\, d\tau = \int K_Y(s,\lambda)\, E(\lambda)\, d\lambda \quad , \tag{1.27}$$

$$E(s) = \int G_Z(s,\tau)\, i(\tau)\, e^{-s\tau}\, d\tau = \int K_{1Z}(s,\tau)\, i(\tau)\, d\tau = \int K_Z(s,\lambda)\, I(\lambda)\, d\lambda \quad . \tag{1.28}$$

C. TRANSFER IMMITTANCE FUNCTIONS

Transfer immittance functions of time-varying linear systems can be defined by direct analogy to time-invarient systems. For example, in time-invarient systems the transfer impedance $Z_{jk}(s)$ is defined by

$$Z_{jk}(s) \doteq \left. \frac{E_j(s)}{I_k(s)} \right|_{I_j(s) = 0 ,\ j \neq k} \quad . \tag{1.29}$$

By direct analogy the following can be defined:

$$G_{Z_{jk}}(s,\tau) = \left.\frac{E_j(s)}{I_k(s)}\right|_{I_j(s) = \begin{cases} e^{-s\tau}, & j = k \\ 0, & j \neq k \end{cases}} , \tag{1.30}$$

$$H_{Z_{jk}}(t,\lambda) = \left.\frac{e_j(t)}{i_k(t)}\right|_{i_j(t) = \begin{cases} e^{\lambda t}, & j = k \\ 0, & j \neq k \end{cases}} , \tag{1.31}$$

$$K_{1Z_{jk}}(s,\tau) = E_j(s)\Big|_{I_j(s) = \begin{cases} e^{-s\tau}, & j = k \\ 0, & j \neq k \end{cases}} , \tag{1.32}$$

and similarly for $k_{Z_{jk}}(t,\tau)$, and for $K_{Z_{jk}}(t,\tau)$. Analogous definitions can also be made on an admittance basis ($G_{Y_{ik}}(s,\tau)$ etc.) or on any mixed-parameter basis.

D. DUALITY CONCEPTS

The introduction of the physical variables, current and voltage, has added a new dimension to duality. First, one can consider duality on the basis of dual-system functions. It was seen before that $G(s,\tau)$, $H(t,\lambda)$; $K_1(s,\tau)$, $K_2(t,\lambda)$ etc. are dual-system functions, and their corresponding arguments are dual arguments. Adding a subscript of Z or Y on the system function does not affect this type of duality, hence $G_Y(s,\tau)$, $H_Y(t,\lambda)$; $K_{1Y}(s,\tau)$, $K_{2Y}(t,\lambda)$ etc. are

duals. To take a dual on the basis of dual-system functions, one replaces all system functions by their duals and uses dual arguments. Equations (1.25) and (1.27), and Equations (1.26) and (1.28) are duals on this basis.

Secondly, one can consider duality on the basis of dual physical quantities such as v and i, Z and Y etc. Using the same system function does not affect duality on this basis, hence $G_Z(s, \tau)$, $G_Y(s, \tau)$; $H_Z(t, \lambda)$, $H_Y(t, \lambda)$ etc. are duals. To take a dual on the basis of dual-physical quantities, one replaces all physical quantities by their duals. On this basis, Equations (1.25) and (1.26) and Equations (1.27) and (1.28) are duals.

Finally one can consider "complete" duality, that is duality on the basis of both physical quantities and system functions. To take a "complete" dual, one replaces physical quantities by their duals, system functions by their dual-system functions, and all arguments by their dual arguments. On this basis, Equations (1.25) and (1.28) and Equations (1.26) and (1.27) are duals.

As a consequence of this extended duality, knowing one relationship is equivalent to knowing four relationships instead of two. For example, given Equation (1.25), one can take its dual on the basis of dual-system function to obtain Equation (1.27); one can take its dual on the basis of physical quantities to obtain Equation (1.26); and one can take its "complete" dual to obtain Equation (1.28).

This concept of system-function duality has a physical significance. For example, a time-invariant system and a time multiplier

are duals, and a time-varying delay system is the dual of a selective frequency-shift system.

II. CASCADED SYSTEMS

A. SYSTEM FUNCTIONS FOR CASCADED SYSTEMS

The impulse-response function $k(t,\tau)$ is useful in finding the over-all system function for a cascade of time-varying systems when the individual system functions are known. For two systems in cascade, as shown in Figure 2.1, the over-all system function $k(t,\tau)$ can be found easily. When the input is an impulse in time applied at time $t = \tau$, the output of the first network in time is

$$y(t) = k^{(1)}(t,\tau) \quad , \tag{2.1}$$

where $k^{(1)}(t,\tau)$ is the impulse response of the first network. Applying the input-output relation of Equation (1.15) to the second system with $y(t)$ as its input yields the over-all impulse-response function:

$$k(t,\tau) = \int_{-\infty}^{\infty} k^{(2)}(t,t_1)\, k^{(1)}(t_1,\tau)\, dt_1 \quad . \tag{2.2}$$

Equation (2.2) is the convolution product for time-varying systems. In a similar manner the over-all mixed impulse response is found to be

$$K_1(s,\tau) = \int_{-\infty}^{\infty} K_1^{(2)}(s,t_1)\, k^{(1)}(t_1,\tau)\, dt_1 \quad . \tag{2.3}$$

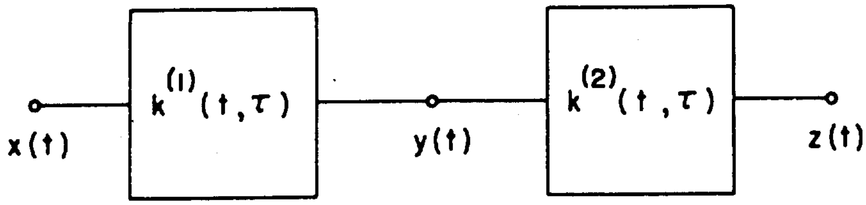


FIGURE 2.1. Two Time-Varying Systems in Cascade.

Equations for the other K-system functions are obtained by taking the duals of Equations (2.2) and (2.3):

$$K(s,\lambda) = \frac{1}{2\pi j}\int_{\Omega} K^{(2)}(s,p_1)\, K^{(1)}(p_1,\lambda)\, dp_1 \quad ,$$

$$K_2(t,\lambda) = \frac{1}{2\pi j}\int_{\Omega} K_2^{(2)}(t,p_1)\, K^{(1)}(p_1,\lambda)\, dp_1 \quad . \tag{2.4}$$

These results are easily extended to a system of n networks in cascade. The over-all $k(t,\tau)$ function is

$$k(t,\tau) = \overbrace{\int\cdots\int}^{n-1} k^{(n)}(t,t_{n-1})\, k^{(n-1)}(t_{n-1},t_{n-2}) \cdots$$

$$k^{(2)}(t_2,t_1)\, k^{(1)}(t_1,\tau)\, dt_1 \cdots dt_{n-1} \quad ; \tag{2.5}$$

the over-all $K_1(s,\tau)$ function is

$$K_1(s,\tau) = \overbrace{\int\cdots\int}^{n-1} K_1^{(n)}(s,t_{n-1})\, k^{(n-1)}(t_{n-1},t_{n-2}) \cdots$$

$$k^{(2)}(t_2,t_1)\, k^{(1)}(t_1,\tau)\, dt_1 \cdots dt_{n-1} \quad . \tag{2.6}$$

The other K system functions can be expressed as the duals of Equations (2.5) and (2.6). Equation (2.5) is not the only expression for the over-all $k(t, \tau)$ function because the convolution may be done over any mixed basis. A more general expression for $k(t,\tau)$ is

$$k(t,\tau) = \left(\frac{1}{2\pi j}\right)^m \overbrace{\int \cdots \int}^{n-1} \gamma^{(n)}(t,\mu_{n-1})\gamma^{(n-1)}(\mu_{n-1}, \mu_{n-2}) \cdots$$

$$\gamma^{(2)}(\mu_2, \mu_1)\ \gamma^{(1)}(\mu_2, \mu_1)\ d\mu_1 \cdots d\mu_{n-1} \quad , \qquad (2.7)$$

where μ_j is either a time or a frequency variable, and $K^{(j)}(\mu_j, \mu_{j-1})$, $K^{(n)}(t, \mu_{n-1})$ and $K^{(1)}(\mu_1,\tau)$ are the K-system functions appropriate to their arguments.

B. THE TERMINATED MULTIPLIER

An important cascaded system is the terminated multiplier. Because of parasitic elements at high frequencies, it becomes necessary to consider multipliers along with their associated input and output networks. The link structure can be regarded as a parallel connection of terminated multipliers; therefore, the analyses presented here are applicable to studying the link structure as a high-frequency modulator. In modulating systems it is important to determine when the system functions are separable into a function of the time variable multiplied by a function of the frequency variable because it is easy to relate the input to its modulated signal when the system function is separable.

The systems of Figure 2.2 and Figure 2.3 are G-separable and H-separable, respectively, with

$$G(s,\tau) = a(\tau)\, B(s) \tag{2.8}$$

for the first system, and

$$H(t,\lambda) = a(t)\, B(\lambda) \tag{2.9}$$

for the second system. The terminated multiplier is shown in Figure 2.4. The over-all system function $K_1(s,\tau)$ is found by considering the

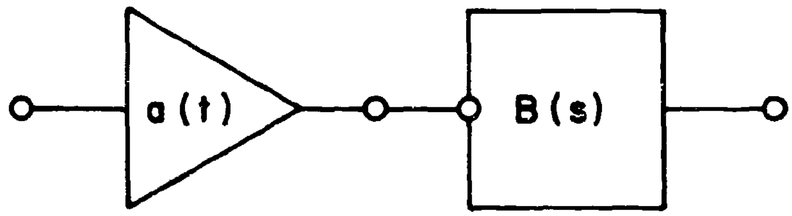


FIGURE 2.2. G-Separable System.

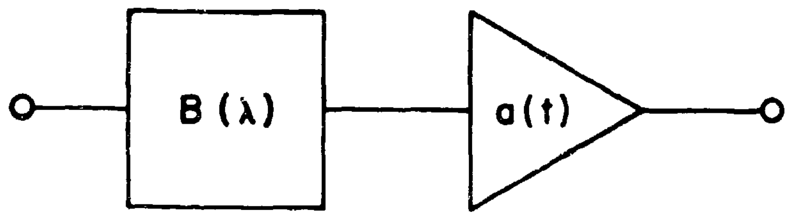


FIGURE 2.3. H-Separable System.

terminated multiplier as the network $B(s)$ in cascade with the G-separable network consisting of the multiplier $a(t)$ followed by the network $C(s)$. Using Equation (2.3) with

$$k^{(1)}(t,\tau) = b(t-\tau) \quad , \tag{2.10}$$

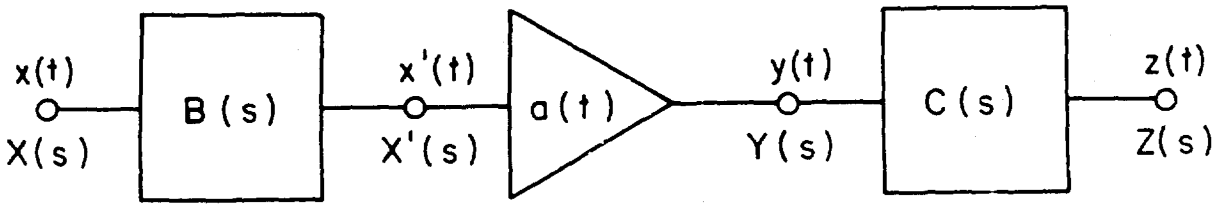


FIGURE 2.4. Multiplier with Its Associated Input and Output Networks.

and

$$K_1^{(2)}(s,\tau) = a(\tau)\, C(s)\, e^{-s\tau} \quad , \tag{2.11}$$

yields

$$K_1(s,\tau) = C(s) \int_{-\infty}^{\infty} a(t)\, e^{-st}\, b(t-\tau)\, dt \quad . \tag{2.12}$$

Making the change of variables

$$r = t - \tau \quad , \tag{2.13}$$

gives

$$G(s,\tau) = C(s) \int_{-\infty}^{\infty} b(r)\, a(r+\tau)\, e^{-sr}\, dr \quad . \tag{2.14}$$

The over-all $K_2(t,\lambda)$ function can be found by considering the terminated multiplier as an H-separable network followed by a network $C(s)$ with

$$K^{(1)}(s,\lambda) = B(\lambda)\, A(s-\lambda) \quad , \tag{2.15}$$

and

$$K_2^{(2)}(t,\lambda) = C(\lambda)\, e^{\lambda t} \quad . \tag{2.16}$$

From Equations (2.4) , (2.15), and (2.16),

$$K_2(t,\lambda) = \frac{B(\lambda)}{2\pi j} \int_{\Omega} A(s-\lambda)\, C(s)\, e^{st}\, ds \quad ; \tag{2.17}$$

making the change of variables,

$$p = s - \lambda \quad , \tag{2.18}$$

yields

$$H(t,\lambda) = \frac{B(\lambda)}{2\pi j} \int_{\Omega} A(p)\, C(p+\lambda)\, e^{pt}\, dp \quad . \tag{2.19}$$

It appears to be difficult to find the general conditions under which the system functions of Equations (2.14) and (2.19) are separable. If, however, one assumes the multiplier to be of exponential form:

$$a(t) = e^{p_o t} \quad , \qquad A(s) = \frac{1}{s - p_o} \quad , \tag{2.20}$$

then

$$\begin{aligned} G(s,\tau) &= B(s-p_o)\, C(s)\, e^{p_o \tau} \\ &= B(s-p_o)\, C(s)\, a(\tau) \quad , \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} H(t,\lambda) &= B(\lambda)\, C(\lambda+p_o)\, e^{p_o t} \\ &= B(\lambda)\, C(\lambda+p_o)\, a(t) \quad . \end{aligned} \tag{2.22}$$

From Equation (2.22), Equation (1.12) and Equation (1.14) one sees that the output of the terminated multiplier for $x(t)$ as the input is the signal $y(t)$ modulated by $a(t) = e^{p_o t}$, where

$$y(t) = \frac{1}{2\pi j} \int_{\Omega} B(\lambda)\, C(\lambda + p_o)\, X(\lambda)\, e^{\lambda t}\, d\lambda \quad . \tag{2.23}$$

Thus, for a separable terminated multiplier structure the input is reshaped according to Equation (2.23) and then is modulated by the multiplier $a(t)$.

C. PERIODIC MULTIPLIERS

Now one can easily extend these results to the class of periodic multipliers $a(t)$, which is indeed an important class of multipliers for the general case of modulation. If $a(t)$ is periodic with period T, it can be expressed in a Fourier series,

$$a(t) = \sum a_i\, e^{p_i t} \quad , \tag{2.24}$$

where

$$p_n = j\,\frac{nT}{2\pi} \quad . \tag{2.25}$$

Now, from Equations (2.21) and (2.22), the system functions become

$$G(s,\tau) = \sum a_i\, B(s - p_i)\, C(s)\, e^{p_i \tau} \quad , \tag{2.26}$$

$$H(t,\lambda) = \sum a_i \; B(\lambda) \; C(\lambda + p_i) \; e^{p_i t} \quad . \tag{2.27}$$

By considering certain particular functions $B(s)$ and $C(s)$, one can make some interesting observations from Equations (2.26) and (2.27). If, for example, $C(\lambda)$ is periodic along all vertical axes with period T, then

$$C(\lambda + p_i) = C(\lambda) \quad ; \tag{2.28}$$

therefore

$$H(t, \lambda) = \sum a_i \; B(\lambda) \; C(\lambda) \; e^{p_i t}$$

$$= B(\lambda) \; C(\lambda) \; a(t) \quad . \tag{2.29}$$

Similarly if $B(s)$ is periodic along all vertical axes with period T,

$$B(\lambda - p_i) = B(\lambda) \quad , \tag{2.30}$$

and

$$G(s,\tau) = B(s) \; C(s) \; a(\tau) \quad . \tag{2.31}$$

The H function of Equation (2.29) and the G function of Equation (2.31) are completely separable.

One can also regard Equations (2.29) and (2.31) in terms of sampling. If, for example, $B(\lambda)$ has a spectrum $B(j\omega)$ along the $j\omega$-axis with magnitude and phase shown in Figure 2.5, with $C(j\omega)$ having the magnitude and phase shown in Figure 2.6, and $a(t)$ a periodic time function such that

$$\omega_o = T \quad , \tag{2.32}$$

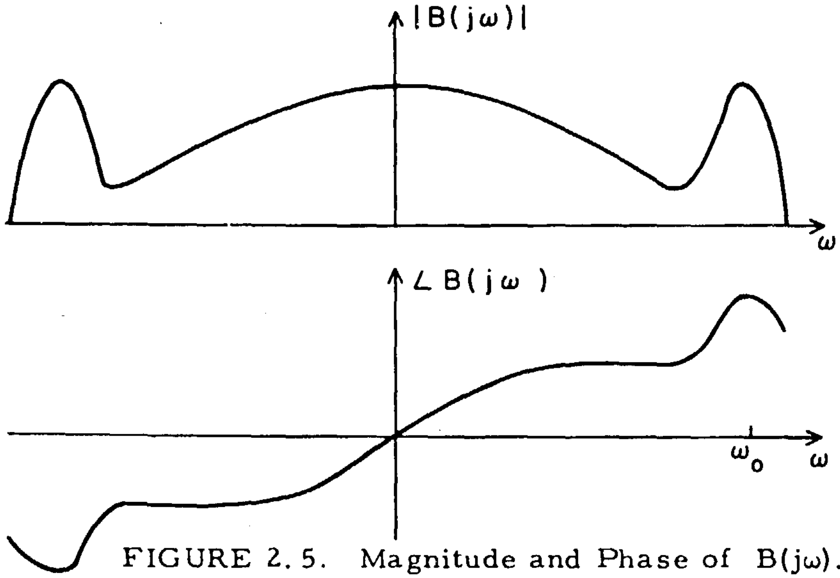


FIGURE 2.5. Magnitude and Phase of $B(j\omega)$.

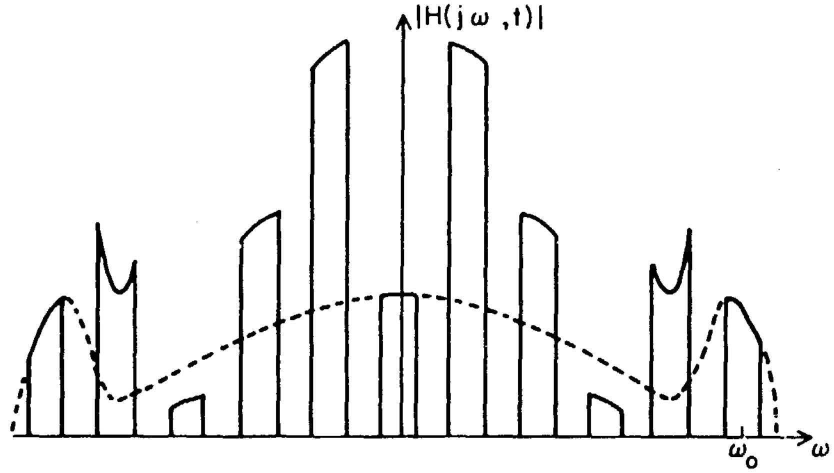


FIGURE 2.7. Magnitude of $H(j\omega, t)$.

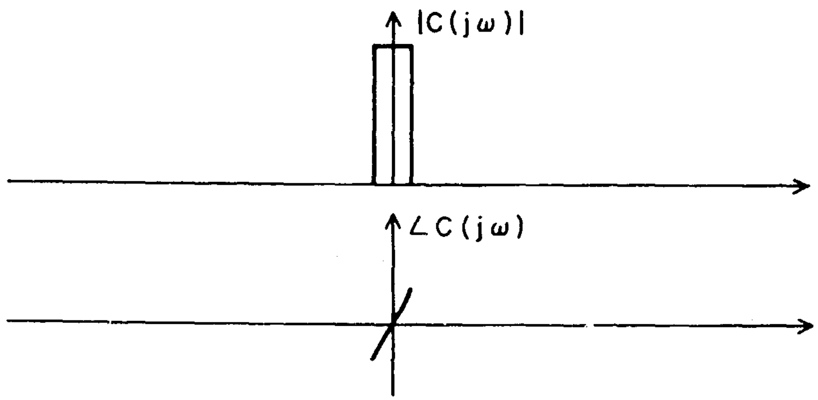


FIGURE 2.6. Magnitude and Phase of $C(j\omega)$.

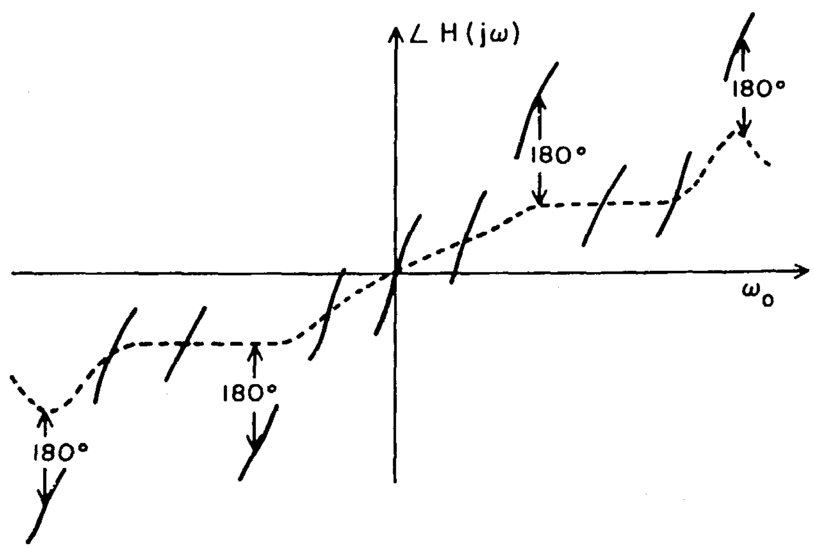


FIGURE 2.8. Phase of $H(j\omega, t)$ at $t = 0$.

$$a_o = 1,\ a_1 = 3,\ a_2 = -2,\ a_3 = \frac{1}{2},\ a_4 = 4,\ a_5 = -1,\ a_k = a_{-k} \quad , \tag{2.33}$$

then $H(t,\lambda)$ has the magnitude curve shown in Figure 2.7.

Note that $|H(t,\lambda)|$ is independent of time. The phase of $H(t,\lambda)$ does, however, vary as a function of time. At $t = 0$, the phase is given in Figure 2.8, and at time t, one must add $nTt/2\pi$ degrees to the piece centered about $n\omega_o/5$.

One can give a geometric interpretation to the phase of $H(j\omega, t)$; i.e., each piece in the curve of Figure 2.8 is traced on the surface of a cylinder of unit radius centered about the ω axis, and a cut is made between each nonzero piece in Figure 2.8. Thus the original cylinder is divided into a number of sections. Now the cylindrical section that contains $n\omega_o/5$ rotates with angular frequency $n\omega_o/5$ radians per second, and the phase angle at $\omega = \omega_1$ is measured in a plane parallel to the x-y plane at $\omega = \omega_1$. The phase of $H(t, j\omega)$, viewed in this way, is shown in Figure 2.9.

Now it is easy to see what would happen for other spectra and values of T. If in the above example, $T > \omega_o$, then $H(t, j\omega)$ would have only one sample, with magnitude and phase shown in Figure 2.10. Note that $H(t, j\omega)$ is not a function of t in this case.

D. LINK STRUCTURE

The methods for finding the system function of a multiplier with its associated input and output networks can be used to analyze the link structure. The link structure can be thought of as having the form

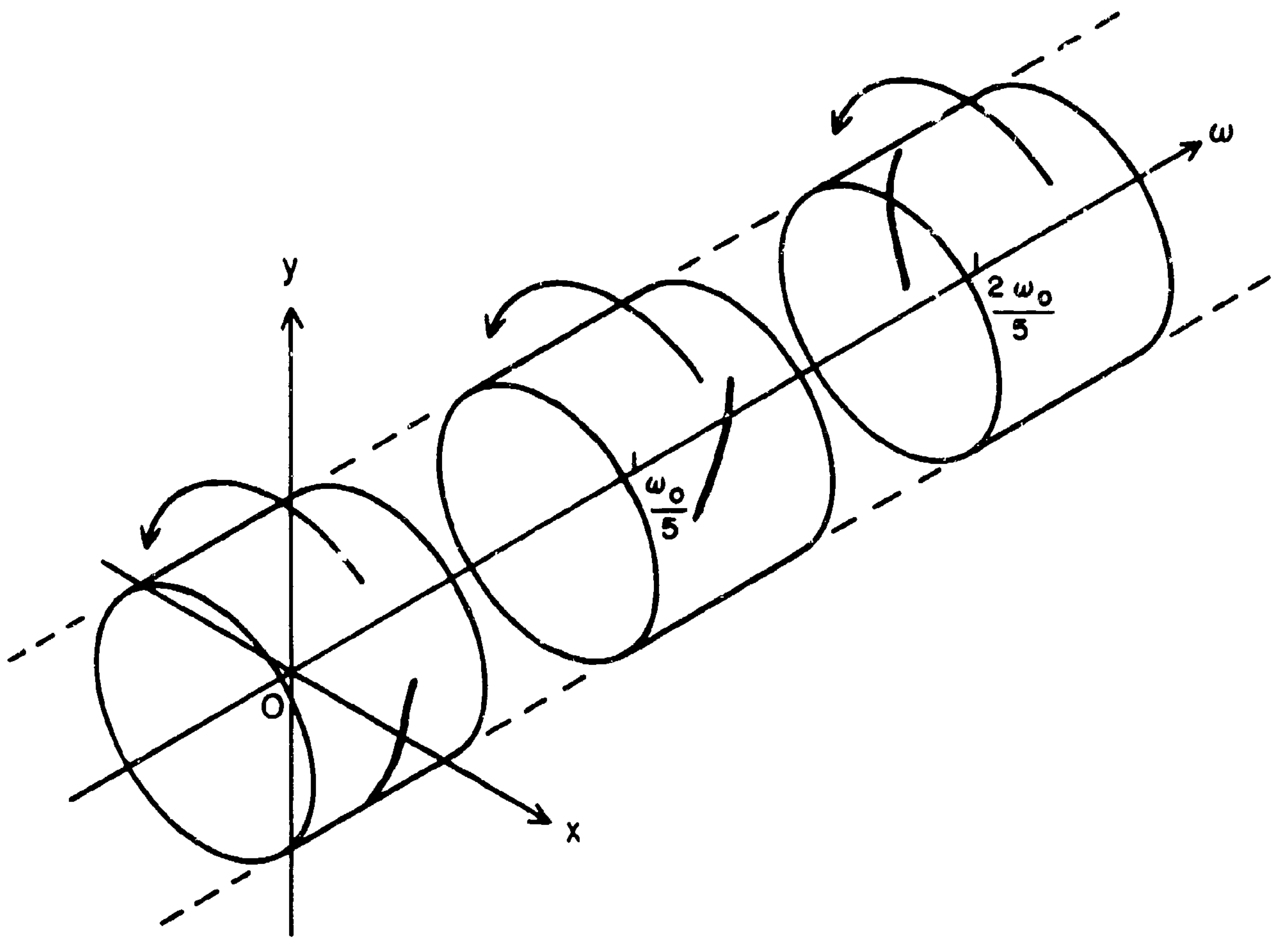


FIGURE 2.9 Phase of $H(t, j\omega)$ Showing Three Sections of Phase Cylinder.

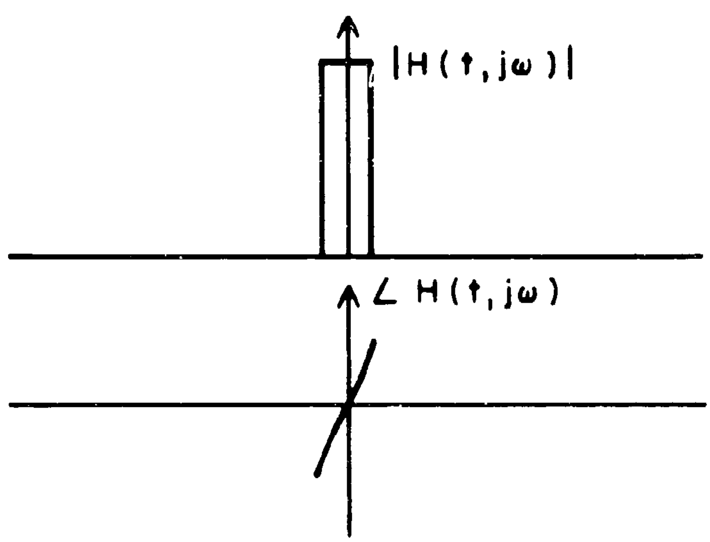


FIGURE 2.10. Magnitude and Phase of $H(t, j\omega)$.

shown in Figure 2.11. If the $a(t)$'s are periodic multipliers, i.e.,

$$a_k(t) = \sum_i a_{ik} e^{p_{ik} t} \quad , \tag{2.34}$$

then

$$H(t,\lambda) = \sum_k \sum_i a_{ik} B_k(\lambda) C_k(\lambda + p_{ik}) e^{p_{ik} t}$$

$$= \sum_k B_k(\lambda) \sum_i a_{ik} C_k(\lambda + p_{ik}) e^{p_{ik} t} \quad , \tag{2.35}$$

and

$$G(s,\tau) = \sum_k C_k(s) \sum_i a_{ik} B_k(s - p_{ik}) e^{p_{ik} \tau} \quad . \tag{2.36}$$

E. ALTERNATING STRUCTURE WITH EXPONENTIAL MULTIPLIERS

The system functions associated with the series alternating structure of Figure 2.12 when the multipliers are exponential,

$$a_i(t) = e^{p_i t} \quad , \tag{2.37}$$

have the separable forms:

$$H(t,\lambda) = B_1(\lambda) B_2(\lambda + p_1) \ldots B_n(\lambda + p_1 + \ldots + p_{n-1}) a_1(t) \ldots a_{n-1}(t) \quad , \tag{2.38}$$

and

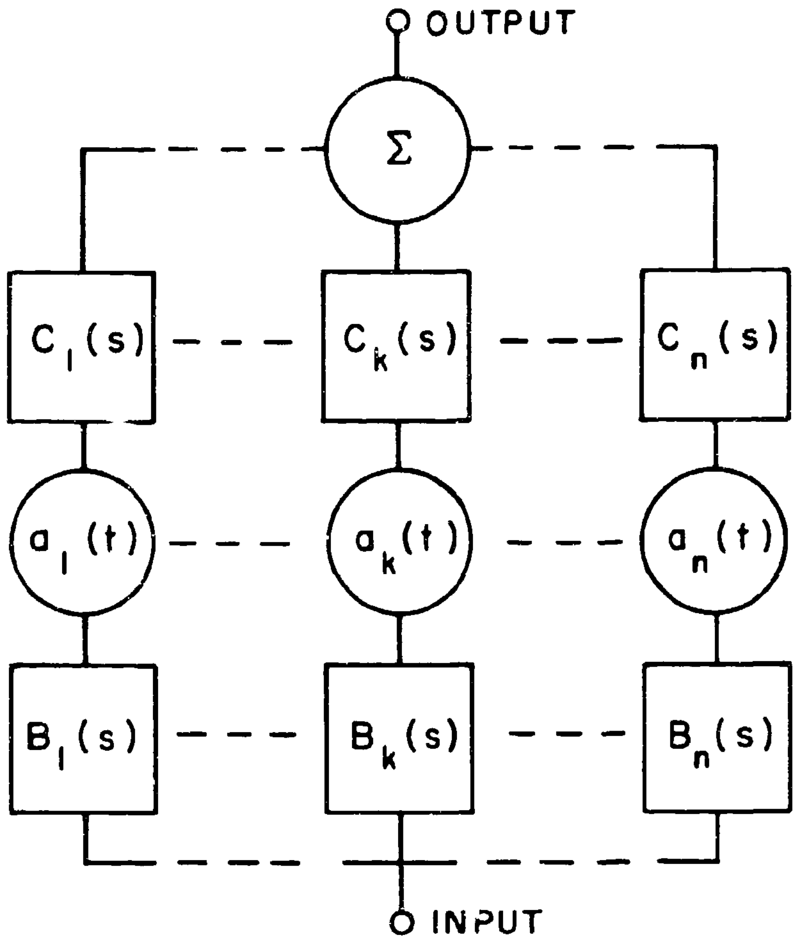


FIGURE 2.11. Link Structure.

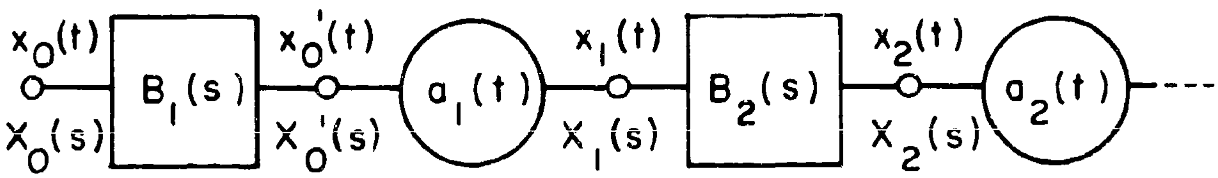


FIGURE 2.12. Series Alternating Structure.

$$G(s,\tau) = B_1(s-p_1-p_2-\dots-p_{n-1})\dots B_{n-1}(s-p_1)\,B_n(s)\;a_1(\tau)\;\dots\;a_{n-1}(\tau) \; . \tag{2.39}$$

Now consider the particular example of the $H(t,\lambda)$ system function for a band-limited input $X(\lambda)$,

$$X(\lambda) = 0 \qquad \text{for} \qquad |\lambda| > \lambda_o \quad . \tag{2.40}$$

Define a restriction of the networks to a band-limited form by

$$B_k(\lambda)\Big|\,\lambda_o = \begin{cases} B_k(\lambda+p_1+\dots+p_{n-1}) & \text{for} \begin{cases} \lambda > -(p_1+\dots+p_{k-1}+\lambda_o) \; , \\ \lambda < \lambda_o-p_1-\dots-p_k-1 \; , \end{cases} \\ 0 & \text{otherwise} \quad . \end{cases} \tag{2.41}$$

Then the quantity of interest for determining the reshaped output time function $x(\lambda)H(t,\lambda)$ becomes

$$X(\lambda)\,H(t,\lambda) = X(\lambda)[B_1(\lambda)\,|\,\lambda_o]\,[B_2(\lambda)\,|\,\lambda_o]\dots[B_n(\lambda)\,|\,\lambda_o]\,e^{(p_1+\dots+p_{k-1})\,t} \quad . \tag{2.42}$$

Now one can note that if

$$B_k(\lambda)\,|\,\lambda_o = \begin{cases} 1 & \text{for} \begin{cases} \lambda > -(p_1+\dots+p_{k-1}+\lambda_o) \; , \\ \lambda < +\lambda_o-p_1-\dots-p_{k-1} \; , \end{cases} \\ 0 & \text{otherwise} \; , \end{cases} \tag{2.43}$$

then

$$X(\lambda)\,H(t,\lambda) = X(\lambda)\,e^{(p_1+\dots+p_{k-1})t} \; , \tag{2.44}$$

and x(t) is not reshaped at all. This means that if $B_k(\lambda)$ is flat λ_o cycles above and below $-(p_1+\dots+p_{k-1})$, then multiplication of the input signal x(t) will result.

If one now considers the multipliers in the alternating structure as periodic functions of time, namely

$$a_i(t) = \sum_{k_i=1}^{n_i} a_{ik_i} e^{p_{ik_i} t} \quad ; \tag{2.45}$$

then by superposition, one immediately obtains

$$H(t,\lambda) = \sum_{k_1 k_2 \dots, k_{n-1}=1}^{n_1 n_2 \dots n_{n-1}} a_{1k_1} \dots a_{n-1,k_{n-1}} B_1(\lambda) B_2\left(\lambda+p_{1k_1}\right) \dots$$
$$B_n\left(\lambda+p_{1k_1}+\dots+p_{n-1,k_{n-1}}\right) e^{\left(p_{1k_1}+\dots+p_{n-1,k_{n-1}}\right)t} \quad , \tag{2.46}$$

and

$$G(s,\tau) = \sum_{k_n \dots k_{n-1}=1}^{n_1 n_2 \dots n_{n-1}} a_{1,k_1} \dots a_{n-1,k_{n-1}} B_1\left(\lambda-p_{1,k_1} \dots p_{n-1,k_{n-1}}\right) \dots$$
$$B_n(\lambda) \; e^{\left(p_{1,k_1}+\dots+p_{n-1,k_{n-1}}\right)\tau} \quad . \tag{2.47}$$

F. MODULATOR-DEMODULATOR STRUCTURE

Note that the alternating structure, which consists of two multipliers and three networks, may be regarded as a model for a modulator-demodulator system, where the middle network embodies the system response of the terminating network of the modulating multiplier, the channel, and the system response of the input network of the demodulating multiplier.

The system responses for the modulator-demodulator structure with multipliers

$$a_1(t) = e^{p_1 t} \quad \text{and} \quad a_2(t) = e^{p_2 t} \tag{2.48}$$

are:

$$H(t, \lambda) = B_1(\lambda)\, B_2(\lambda + p_1)\, B_3(\lambda + p_1 + p_2)\, e^{(p_1 + p_2)\, t} \quad , \tag{2.49}$$

$$G(s, \tau) = B_1(s - p_1 - p_3)\, B_2(s - p_1)\, B_3(s)\, e^{(p_1 + p_2)\tau} \quad . \tag{2.50}$$

For $p_1 = -p_2$, and the input limited to the band $-\lambda_o < \lambda < \lambda_o$ and the conditions:

$B_1(\lambda)$ and $B_3(\lambda)$ flat for λ_o cycles above and below 0 cycles, and

$B_2(\lambda)$ flat for λ_o cycles above and below p_1 cycles, then

$$X(\lambda)\, H(t, \lambda) = X(\lambda) \quad , \tag{2.51}$$

and the input to the second multiplier is

$$X(\lambda)\, e^{p_1 t} \qquad .$$

Thus, the input signal has been modulated and demodulated successfully.

For the case of multipliers that are real sinusoids, i.e.,

$$a_1(t) = e^{p_1 t} + e^{p_2 t} \qquad ,$$

$$a_2(t) = e^{p_2 t} + e^{-p_2 t} \qquad , \tag{2.52}$$

one obtains, by using Equation (2.38),

$$\begin{aligned} H(t,\lambda) = \; & B_1(\lambda)\, B_2(\lambda + p_1)\, B_3(\lambda + p_1 + p_2)\, e^{(p_1 + p_2)t} \\ & + B_1(\lambda)\, B_2(\lambda + p_1)\, B_3(\lambda + p_1 - p_2)\, e^{(p_1 - p_2)t} \\ & + B_1(\lambda)\, B_2(\lambda - p_1)\, B_3(\lambda + p_2 - p_1)\, e^{(p_2 - p_1)t} \\ & + B_1(\lambda)\, B_2(\lambda - p_1)\, B_3(\lambda - p_1 - p_2)\, e^{-(p_1 + p_2)t} \qquad . \end{aligned} \tag{2.53}$$

If $p_1 = p_2$ and the networks satisfy the conditions in the previous example, then

$$X(\lambda)\, H(t,\lambda) = 2\, X(\lambda) \qquad , \tag{2.54}$$

and the input to the second multiplier is

$$X(\lambda)\left(e^{p_1 t} + e^{p_2 t}\right) = X(\lambda)\, a_1(t) \quad . \tag{2.55}$$

Thus, the input signal has been modulated and demodulated successfully.

III. REPRESENTATION OF SYSTEM FUNCTIONS BY SAMPLING AND SERIES EXPANSIONS

A. INTRODUCTION

The expansion of a function of several variables in a series,

$$f(x_1, x_2, \ldots, x_N) = \sum_{i=1} a_i g_i(x_1, x_2, \ldots, x_N) \quad , \qquad (3.1)$$

that is convergent uniformly or in the mean is useful for theoretical calculations as well as in practical applications where one can store a close approximation to a continuous function by storing a finite number of expansion coefficients.

When the expansion coefficients are samples of the function, the expansion is a sampling series and the functions $g_i(\vec{x})$ can be considered as interpolation functions. The extension of the sampling theorem in one dimension to N Euclidean dimensions by Petersen and Middleton[6] is reviewed in Appendix A. Applications to expansions of system functions will be detailed in this chapter.

Given that the sets

$$\left\{\phi_i^{(j)}\right\}_{i=1,2,3,\ldots} \quad , \quad \text{for } j = 1, 2, \ldots, N \quad ,$$

are complete sets on the intervals (a_j, b_j), respectively, then $f(x_1, x_2, \ldots, x_N)$ can be expanded[7] in a mean-square convergent series in terms of the set

$$g_i(x_1, x_2, \ldots, x_N)_{i=1,2,\ldots} = \left\{\phi_{i_1}^{(1)}(x_1)\phi_{i_2}^{(2)}(x_2)\ldots\phi_{i_N}^{(N)}(x_N)\right\}_{i_1, i_2, \ldots i_N = 1, 2, \ldots} \tag{3.2}$$

For $N = 2$ the system functions can be expanded in a double series, and if the sets

$$\left\{\phi_i^{(j)}\right\}_{i=1,2,\ldots}$$

are orthonormal for $j = 1, 2,$ the expansion coefficients can be evaluated by a network that is quite similar to the network used to evaluate the samples for the two dimensional sampling series. If the system function $h(t, v)$ is expanded in terms of the set

$$\left\{\phi_i(t)\, \psi_j(v)\right\}_{i,j=1,2,\ldots} ,$$

where $\psi_j(t)$ is realizable as the impulse response of a passive network, then the truncated expansion of $h(t, v)$ has a realization as a parallel combination of H-separable networks.

B. SAMPLING EXPANSIONS

The one dimensional sampling theorem in time states that for a function $f(t)$, whose Fourier transform $F(\omega)$ is zero outside the band $|\omega| \leq 2\pi B$, one can specify sampling times and an interpolation function $g(t)$ such that

$$f(t) = \sum_{-\infty}^{\infty} f(t_k)\, g(t-t_k) \quad . \tag{3.3}$$

When the samples are taken as the slowest rate $(t_k = k/2B)$, $g(t)$ is the cardinal weighting function:

$$g(t) = \frac{\sin 2\pi Bt}{2\pi Bt} \quad . \tag{3.4}$$

The one dimensional sampling theorem in frequency states that a function $F(\omega)$, whose inverse transform $f(t)$ is time-limited, can be expanded in a sampling series.

The proof of the sampling theorem in N dimensions is reviewed in Appendix A. In particular, the two dimensional case is of interest for system functions. The theorem in two dimensions states that for a function $\gamma(s, r)$ (where s or r may be either time or frequency variables), whose double Fourier transform $\gamma(u, v)$ is zero outside a finite region of the (u, v) plane, one can find sampling times s_k and r_k and an interpolation function $g(s, r)$ such that

$$\gamma(s, r) = \sum_{n, m=-\infty}^{\infty} \gamma(s_n, r_m)\, g(s-s_n, r-r_m) \quad . \tag{3.5}$$

The sampling times and interpolation function have a degree of arbitrariness depending on the region in the (u, v) plane, outside of which γ is zero. For the simplest case the region is a rectangle: $R=(-s_o, s_o) \times (-r_o, r_o)$, and the interpolation function $g(s, r)$ is a product of cardinal

interpolation functions:

$$g(s, r) = \frac{\sin s_o s}{s_o s} \cdot \frac{\sin r_o r}{r_o r} \quad . \tag{3.6}$$

Thus, the system function $k(t, \tau)$, whose double Fourier transform $K(\omega, \zeta)$ is zero outside the rectangle R in the (ω, ζ) plane, where

$$R = (-2\pi B,\ 2\pi B) \times (-2\pi W, 2\pi W) \quad , \tag{3.7}$$

can be expanded in the series

$$k(t, \tau) = \sum_{n, m=-\infty}^{\infty} k\left(\frac{n}{2B}, \frac{m}{2W}\right) \frac{\sin 2\pi B\left(t - \frac{n}{2B}\right)}{2\pi B\left(t - \frac{n}{2B}\right)} \frac{\sin 2\pi W\left(\tau - \frac{m}{2W}\right)}{2\pi W\left(\tau - \frac{m}{2W}\right)} \quad , \tag{3.8}$$

where because of causality

$$k\left(\frac{n}{2B}, \frac{m}{2W}\right) = 0 \quad , \quad \text{for } \frac{m}{2W} > \frac{n}{2B} \quad . \tag{3.9}$$

The system function $K_1(\omega, \tau)$ whose double Fourier transform $K_2(t, \zeta)$ is zero outside the rectangle in the (t, ζ) plane,

$$R = (-T_o, T_o) \times (-2\pi B, 2\pi B) \quad , \tag{3.10}$$

can be expanded in the sampling series

$$K_1(\omega, \tau) = \sum_{j, k=-\infty}^{\infty} K_1\left(\frac{\pi j}{T_o}, \frac{k}{2B}\right) \frac{\sin T_o\left(\omega - \frac{\pi k}{T_o}\right)}{T_o\left(\omega - \frac{\pi k}{T_o}\right)} \frac{\sin 2\pi B\left(\tau - \frac{k}{2B}\right)}{2\pi B\left(\tau - \frac{k}{2B}\right)} \quad . \tag{3.11}$$

The system functions $K_2(t, \zeta)$ or $K(\omega, \zeta)$ can be expanded by the sampling theorem if their double transforms $K_1(\omega, \tau)$ or $k(t, \tau)$ vanish outside of a finite portion of the (ω, τ) or (t, τ) planes, respectively. The four possible expansions, depending on whether the system is time-limited in both variables, frequency-limited in both variables, or time-limited in one variable and frequency-limited in the other, are to be expected by time-frequency duality.[5]

To instrument the expansion of $k(t, \tau)$ by the sampling series given in Equation (3.8), one must take an infinite number of samples. Since this may be infeasible from a practical point of view, one could take a finite, but large enough number of samples inside a certain finite, but possibly large region of the (t, τ) plane with the samples outside this region small enough so that the truncated sum differs from $k(t, \tau)$ by an acceptable amount of error. For simplicity, take the region to be the square $S = (-T_o, T_o) \times (-T_o, T_o)$ in the (t, τ) plane. To evaluate the sample values

$$k\left(\frac{n}{2B}, \frac{m}{2W}\right) \quad \text{for} \quad n = \left[m\,\frac{2B}{2W}\right], \left[m\,\frac{2B}{2W}\right] + 1, \ldots, \left[2BT_o\right] \qquad ,$$

(where the symbol $[a]$ means the largest integer contained in the number a), one applies an impulse to the system at time $m/2W$ and then samples the output at the times $n/2B$, for $n > [m2B/2W]$, by multiplying the output by impulses at the times $n/2B$ and then integrating. The instrumentation of this scheme is shown in Figure 3. 1.

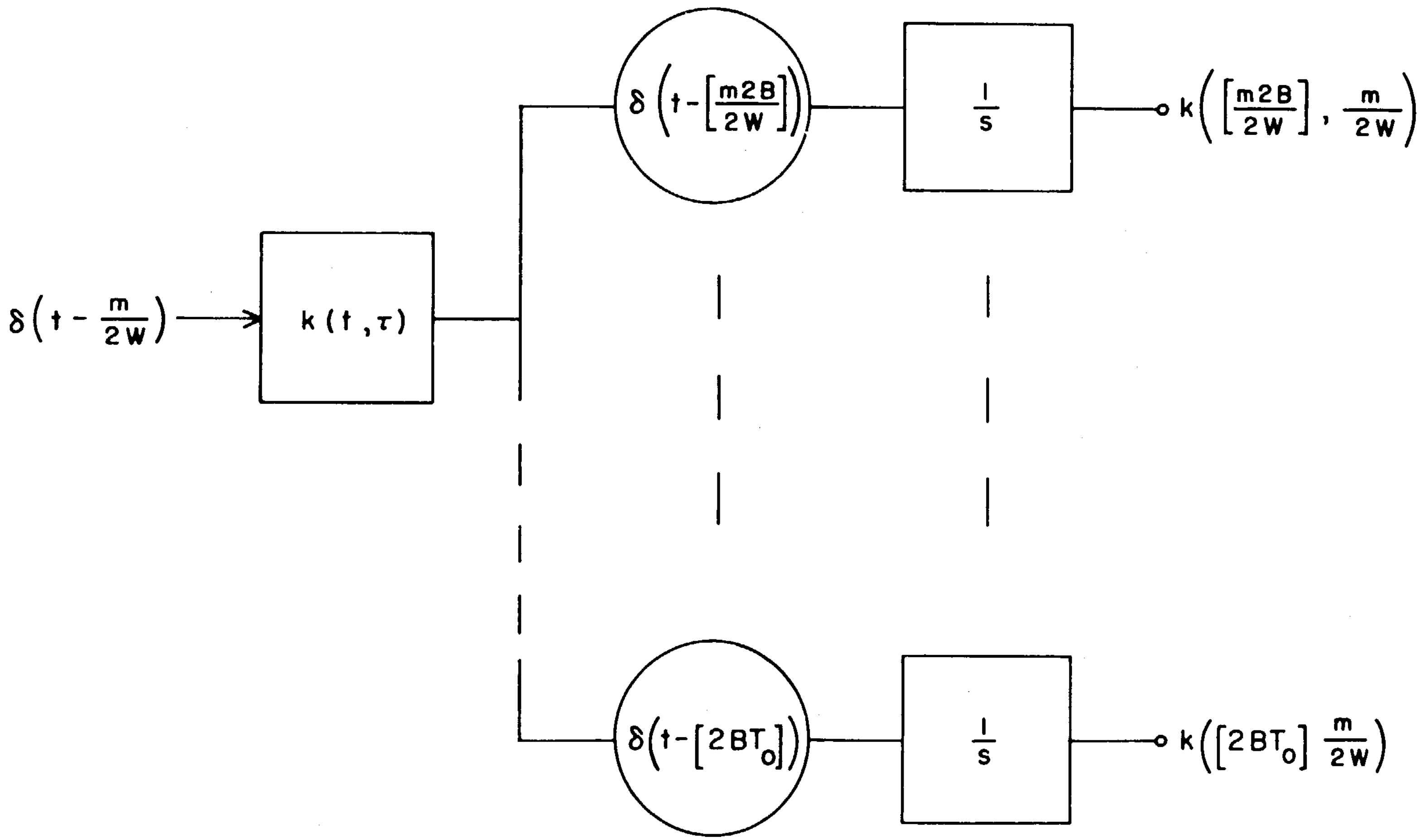


FIGURE 3.1. Coefficient Evaluation Scheme for Sampling Series.

In order to evaluate all the sample values in the square S, one must let m vary over the range $\left(-[2WT_o], -[2WT_o]+1, \ldots, [2WT_o]\right)$. To do this there must be either $[4WT_o]$ identical systems with $\delta(t-m/2W)$ applied to the m^{th} system whose output is sampled as in Figure 3.1, or the time variation must be controlled by some variable elements that can be "reset" to zero, preparing the system to evaluate the next set of coefficients.

Implementation of the expansion of $K_1(\omega, \tau)$ can be done in a manner that is easy to instrument. Evaluation of the sample value $K_1(\pi n/T_o, m/2B)$ for $m = -M, -M+1, \ldots, M$ is done by applying an impulse in frequency $\delta(\omega - \pi n/T_o)$ to the network and then sampling the output in time at the times $m/2B$. Thus, one applies a complex sinusoid of frequency $\pi n/T_o$ and samples the output in time. This technique is especially useful in measuring channels such as the atmosphere. The evaluation process must be repeated for each n to obtain all the samples.

C. EXPANSION OF THE SYSTEM FUNCTIONS IN A DOUBLE SERIES

1. Time-Invariant Systems

For time-invariant systems, the impulse response function $h(t)$ can be expanded over the interval (a, b) in the series

$$h(t) = \sum_{i=1}^{\infty} a_i \, \phi_i(t) \qquad , \tag{3.12}$$

where

$$\left\{\phi_i(x)\right\}_{i=1,2,\ldots}$$

is a complete orthonormal set over the interval (a, b), and where the coefficient a_i is given by

$$a_i = \int_a^b h(t)\ \phi_i(t)\ dt \qquad . \qquad (3.13)$$

For $(a, b) = (0, \infty)$, if $\phi_i(t)$ can be realized as the impulse response of a network, then a_i can be evaluated according to the following scheme, suggested by Huggins:[8] The signal $h(-t)$ is applied to the input of each of the networks with impulse response $\phi_i(t)$, and the output sampled at time zero is the desired coefficient. The system shown in Figure 3.2 is used to implement this scheme.

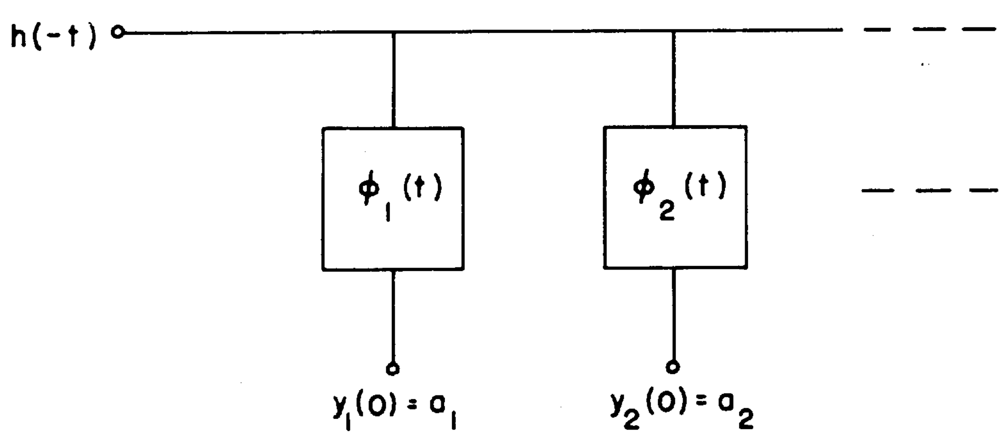


FIGURE 3.2. Coefficient Evaluation Scheme for Single Series.

Practical restrictions may allow one to reverse only T seconds of $h(t)$, in which case the coefficients obtained by the scheme of Figure 3.2 are

$$y_i(0) = \int_0^T h(t)\, \phi_i(t)\, dt \qquad . \tag{3.14}$$

These $y_i(0)$ are an approximation to a_i, if $h(t)$ does not vanish for $t > T$. This same approximation to a_i may also be obtained by applying $h(t)$ to the network with impulse response $\phi_i(T-t)\, u(t)$ and then sampling the output at time T. Another way to evaluate the coefficient a_i is to sample the output of the given network $h(t)$ at time zero when the input is $\phi_i(-t)$. This process must be repeated for each i, unless there are as many identical networks as coefficients to evaluate.

2. Time-Varying Systems

Consider the sets of orthonormal functions

$$\left\{\phi_i(x)\right\}_{i=1,2,\ldots} \qquad \text{and} \qquad \left\{\psi_j(y)\right\}_{j=1,2,\ldots} \qquad ,$$

which are complete on the intervals (a, b) and (c, d), respectively. Their Cartesian product set

$$\left\{\phi_i(x)\, \psi_j(y)\right\}_{i,j=1,2,\ldots}$$

is complete on the rectangle R $(a, b) \times (c, d)$. If the set $\left\{\phi_i\right\}$ is complete on (c, d) as well as on (a, b), then the set $\left\{\phi_i(x)\, \phi_j(y)\right\}$ is

complete on R. The system function $k(t, \tau)$ can be expanded in a double series that is valid over R:

$$k(t, \tau) = \sum_{n, m=1}^{\infty} a_{nm} \, \phi_n(t) \, \psi_m(\tau) \qquad , \tag{3.15}$$

where a_{nm} is given by

$$a_{nm} = \iint\limits_{\substack{(t, \tau) \in R \\ t > \tau}} \phi_n(t) \, k(t, \tau) \, \psi_m(\tau) \; dt \; d\tau \qquad . \tag{3.16}$$

Equation (3.16) is valid for all (t, τ) if $k(t, \tau)$ vanishes outside of R or if R is the entire (t, τ) plane. The coefficient a_{nm} can be evaluated by means of a network with the input $\psi_m(t)$ and where the integration is performed over the limits (a, b), as shown in Figure 3.3a. It is easily seen that the output of this network is indeed a_{nm} and the set of coefficients

$$\left\{a_{nm}\right\} \quad n = 1, 2, \ldots, N$$

can be evaluated simultaneously by the network shown in Figure 3.4. As in the instrumentation of the two dimensional sampling theorem, to let m vary from 1 to M, one must either have M identical networks or be able to "reset" the network. Note the similarity between the coefficient evaluation scheme of Figure 3.4 and the sample evaluation network of Figure 3.1. They are exactly the same when

$$\left\{\phi_i(t)\right\} \qquad \left\{\psi_i(t)\right\} \qquad \left\{\delta(t - iT)\right\} \qquad . \tag{3.17}$$

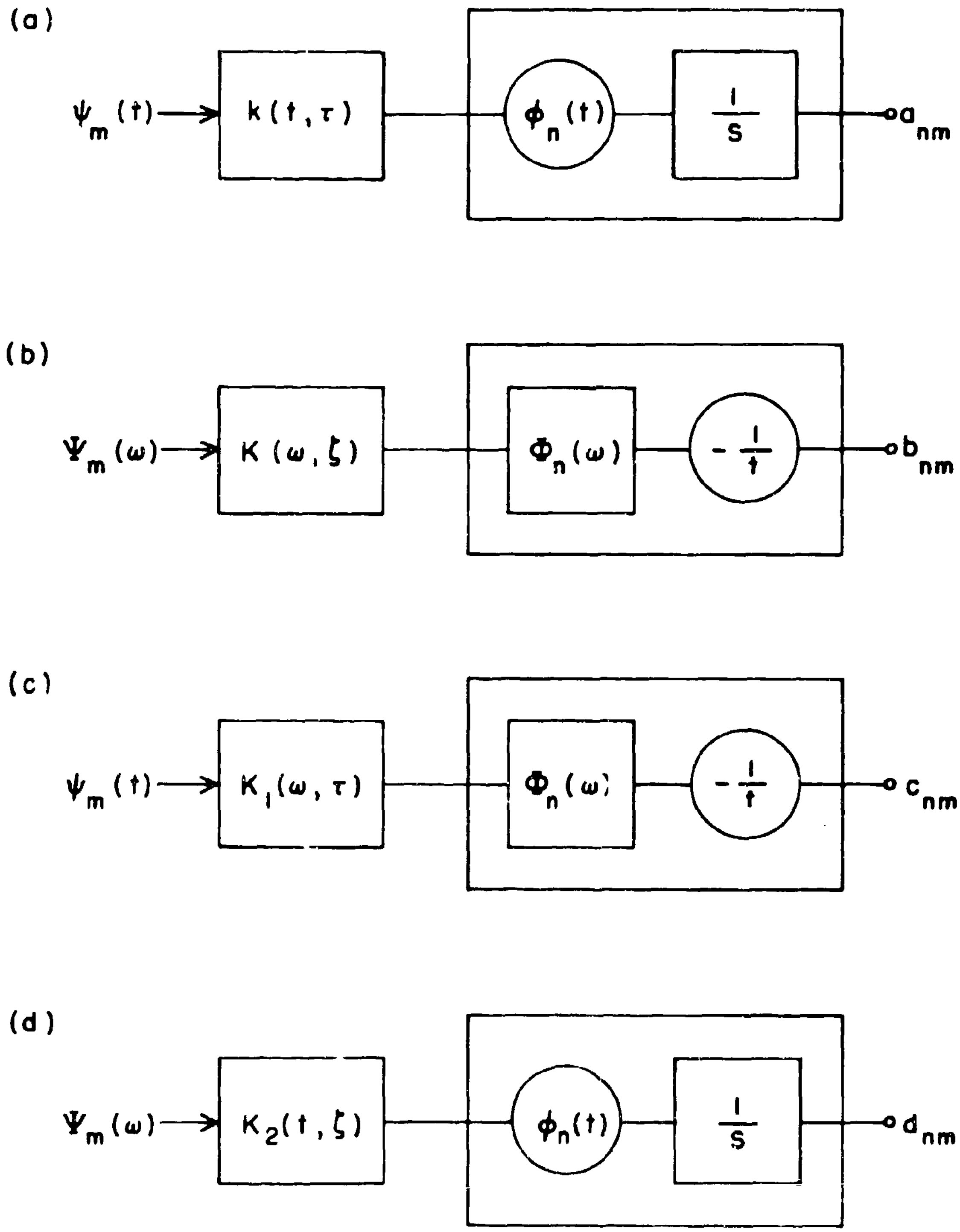


FIGURE 3.3. Coefficient Evaluation Scheme for a) $k(t, \tau)$; b) $K(\omega, \zeta)$; c) $K_1(\omega, \tau)$, d) $K_2(t, \zeta)$.

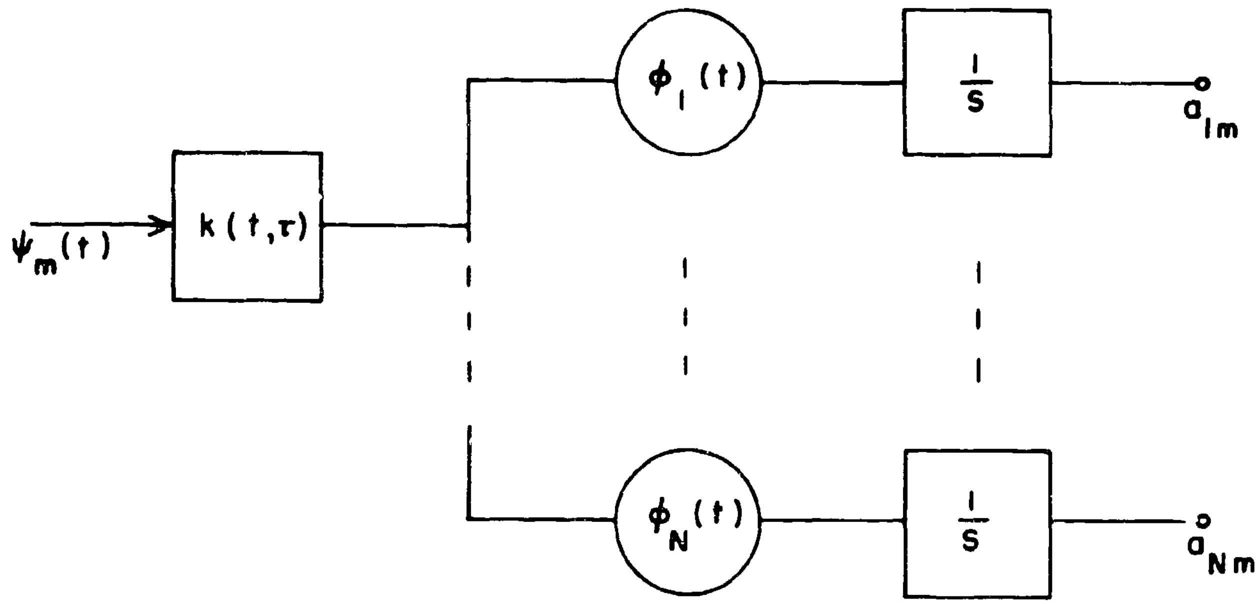


FIGURE 3.4. Coefficient Evaluation Scheme for Double Series.

The sets in Equation (3.17) are not complete in L^2, but the sets

$$\left\{\phi_i(t)\right\} \qquad \left\{\psi_i(t)\right\} \qquad \left\{\frac{\sin 2\pi B\left(t - \frac{i}{2B}\right)}{2\pi B\left(t - \frac{i}{2B}\right)}\right\} \tag{3.18}$$

are complete on $(-\infty, \infty)$ for functions which are appropriately frequency-limited. The coefficients obtained by using the sets of Equation (3.18) in the scheme of Figure 3.4 are the same as those obtained by using the sets of Equation (3.17) in the scheme of Figure 3.4.

Since the system $k(t,\tau)$ is causal, the truncated expansion of Equation (3.15) can be improved by restricting it to be zero for $t < \tau$:

$$k(t,\tau) \approx \begin{cases} \sum_{n,m=1}^{N,M} a_{nm} \phi_n(t)\ \psi_m(\tau) & \text{for } (t,\tau) \in R \text{ and } t > \tau , \\ 0 & \text{for } t < \tau . \end{cases} \tag{3.19}$$

Where $\Phi_i(\omega)$ and $\Psi_i(\omega)$ are defined by the Fourier transforms

$$\Phi_i(\omega) = \int_{-\infty}^{\infty} \phi_i(t)\, e^{-j\omega t}\, dt ,$$

$$\Psi_i(\omega) = \int_{-\infty}^{\infty} \psi_i(t)\, e^{-j\omega t}\, dt , \tag{3.20}$$

the sets $\left\langle \Phi_i(\omega) \right\rangle$ and $\left\langle \Psi_i(\omega) \right\rangle$ are orthonormal and complete over the intervals (a', b') and (c', d') that are appropriate for the intervals (a, b) and (c, d), respectively.

$K(\omega, \zeta)$, the dual system function to $k(t,\tau)$, can be expanded on the interval $R' = (a', b') \times (c', d')$ in terms of the complete orthonormal set $\left\langle \Phi_i(\omega)\ \Psi_j(\zeta) \right\rangle$:

$$K(\omega, \zeta) = \sum_{n,m=1}^{\infty} b_{nm}\, \Phi_n(\omega)\ \Psi_m(\zeta) , \tag{3.21}$$

where

$$b_{nm} = \iint_{(\omega,\zeta) \in R'} \Phi_n(\omega)\ K(\omega, \zeta)\ \Psi_m(\zeta)\, d\omega\, d\zeta . \tag{3.22}$$

If $K(\omega, \zeta)$ is zero outside of R', or if R' is the entire (ω, ζ) plane, then Equation (3.21) is valid on the entire (ω, ζ) plane. The coefficient b_{nm} can be evaluated by the system shown in Figure 3.3b. The systems of Figures 3.3a and 3.3b are duals, as one would expect from time-frequency duality.

The system function $K_1(\omega, \tau)$ can be expanded in terms of the complete orthonormal set $\left\{\Phi_j(\omega)\ \psi_k(\tau)\right\}$ on the rectangle $R_1 = (a', b') \times (c, d)$ by

$$K_1(\omega, \tau) = \sum_{n, m=1}^{\infty} c_{nm}\, \Phi_n(\omega)\, \psi_m(\tau) \quad , \tag{3.23}$$

where

$$c_{nm} = \iint_{(\omega, \tau) \in R_1} \Phi_n(\omega)\, K_1(\omega, \tau)\, \psi_m(\tau)\, d\omega\, d\tau \quad . \tag{3.24}$$

Similarly $K_2(t, \zeta)$ can be expanded in terms of the set $\left\{\phi_j(t)\ \Psi_k(\zeta)\right\}$, which is complete and orthonormal on $R_2 = (a, b) \times (c', d')$, by

$$K_2(t, \zeta) = \sum_{n, m=1}^{\infty} d_{nm}\, \phi_n(t)\, \Psi_m(\zeta) \quad , \tag{3.25}$$

where

$$d_{nm} = \iint_{(t, \zeta) \in R_2} \phi_n(t)\, K_2(t, \zeta)\, \Psi_m(\zeta)\, dt\, d\zeta \quad . \tag{3.26}$$

The coefficients c_{nm} and d_{nm} can be evaluated by the dual networks shown in Figure 3.3c and Figure 3.3d, respectively.

It is important to notice that for a given system, the single Fourier transform of the expansion of $k(t, \tau)$ given by Equation (3.15) is of the form of the expansions of Equation (3.23) or Equation (3.25), and the double transform of $k(t, \tau)$ is of the form of Equation (3.21); hence,

$$a_{nm} = b_{nm} = c_{nm} = d_{nm} \quad , \tag{3.27}$$

and the outputs of the four systems of Figure 3.3 are the same.

By considering two dimensional systems such as optical systems, radar systems, or acoustical systems, one may obtain a coefficient evaluation scheme that is the direct two dimensional analogy of the scheme introduced by Huggins.[8] The input-output relation for a two dimensional system is

$$y(p_1, p_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(p_1, p_2, u_1, u_2)\ x(u_1, u_2) du_1\, du_2 \quad , \tag{3.28}$$

where $k(p_1, p_2, u_1, u_2)$ gives the output at coordinates (p_1, p_2) due to an impulse at the coordinates (u_1, u_2). The coefficient a_{nm} is the output at the coordinates (n, m) of the system

$$k(p_1, p_2, u_1, u_2) = \sum_{n, m=1} \phi_n(u_1)\, \psi_m(u_2)\ \frac{\sin(p_1 - n)}{(p_1 - n)}\ \frac{\sin(p_2 - m)}{(p_2 - m)} \tag{3.29}$$

when the input is $k(t, \tau)$.

The choice of the sets $\langle \phi_i \rangle$ and $\langle \psi_i \rangle$ for a given function of two variables is important, in that a judicious choice may give a truncated expansion that has fewer terms for a given error. For example, a degenerate kernel such as $\sin x \sin y$ on $(0, 2\pi) \times (0, 2\pi)$ should be recognized as its own expansion in terms of the sets

$$\langle \phi_k(x) \rangle = \langle \psi_k(x) \rangle = \langle \sin kx, \ \cos kx \rangle \quad . \tag{3.30}$$

If the complete sets are eigenfunctions of the kernel $k(t, \tau)$, then the expansion contains no crossterms. From the kernel $k(t, \tau)$, one can construct the symmetric kernels

$$k'(t, \tau) = \int_c^d k(t, u)\, k(\tau, u)\, du \quad ,$$

$$k''(t, \tau) = \int_a^b k(u, t)\, k(u, \tau)\, du \quad . \tag{3.31}$$

There exists a sequence of pairs of eigenfunctions and associated eigenvalues such that

$$\phi_v(t) = \lambda_v \int_c^d k(t, \tau) \ \psi_v(\tau)\, d\tau \quad ,$$

$$\psi_v(t) = \lambda_v \int_a^b k(\tau, t)\, \phi_v(\tau)\, d\tau \quad ,$$

$$\phi_v(t) = \lambda_v^2 \int_a^b k'(t, \tau)\, \phi_v(\tau)\, d\tau \quad ,$$

$$\psi_v(t) = \lambda_v^2 \int_c^d k''(t,\tau)\, \psi_v(\tau)\, d\tau \quad . \tag{3. 32}$$

The sets $\left\langle \phi_v \right\rangle$ and $\left\langle \psi_v \right\rangle$ are complete; the coefficient of the expansion of $k(t, \tau)$ is

$$a_{nm} = \int_a^b \phi_n(t) \left(\int_c^d k(t,\tau)\, \psi_m(\tau) d\tau \right) dt = \int_a^b \frac{\phi_n(t)\phi_m(t)}{\lambda_m} dt = \frac{\delta_{nm}}{\lambda_m} \quad , \tag{3. 33}$$

and

$$k(t,\tau) = \sum_{m=1}^{\infty} \frac{\phi_m(t)\, \psi_m(\tau)}{\lambda_m} \quad . \tag{3. 34}$$

Furthermore, if the right-hand side converges uniformly, then it converges uniformly to $k(t,\tau)$.

In the case where $k(t, \tau)$ is symmetric, $\phi_v = \psi_v$. If the eigen values are positive, the expansion

$$k(t,\tau) = \sum_{m=1}^{\infty} \frac{\phi_m(t)\, \phi_m(\tau)}{\lambda_m} \tag{3. 35}$$

converges uniformly in both variables, according to Mercer's theorem.

D. SYNTHESIS BY DOUBLE EXPANSION

Consider the double expansion of the system function $h(t, v)$ in terms of the complete sets $\left\langle \phi_i \right\rangle$ and $\left\langle \psi_i \right\rangle$,

$$h(t, v) = \sum_{n, m=1}^{\infty} a_{nm}\, \phi_n(t)\, \psi_m(v) \quad . \tag{3. 36}$$

This expansion is valid on a rectangle R where $\psi_m(t)$ is realizable as the impulse response of a time-invariant system. The truncated expansion is realizable as a parallel combination of H-separable networks in which a typical branch consists of a network of impulse response $\psi_m(t)$ followed by a multiplier $a_{nm}\,\phi_n(t)$.

For example, consider the particular system function

$$k(t,\tau) = \begin{cases} 0 & t < \tau \quad \text{or} \quad t < 0 \ , \\ S(t-2\tau) & \text{otherwise} \ , \end{cases} \tag{3.37}$$

where

$$S(x) = \frac{1}{\Delta}\left\{u_{-1}(x) - u_{-1}(x-\Delta)\right\} \ . \tag{3.38}$$

For small Δ this system function is an approximation of a time-varying delay system with delay τ. The range $(0,T) \times (0,T)$ of the $k(t,\tau)$ function corresponds to the range $R = (0,T) \times (0,T/2)$ of the $h(t,v)$ function, where

$$h(t,v) = \begin{cases} 0 & v < 0 \quad \text{or} \quad t < 0 \ , \\ S(2v-t) & \text{otherwise} \ . \end{cases} \tag{3.39}$$

Figure 3.5 shows a plot of the $h(t,v)$ function of Equation (3.39). Take the complete sets on the ranges $(0,T)$ and $(0,T/2)$, respectively, to be

$$\left\{\phi_k\right\} = \left\{\sqrt{\frac{1}{T}}\, e^{j\frac{2\pi k}{T}t}\right\}_{k=0,\pm 1,\pm 2,\ldots} \ ,$$

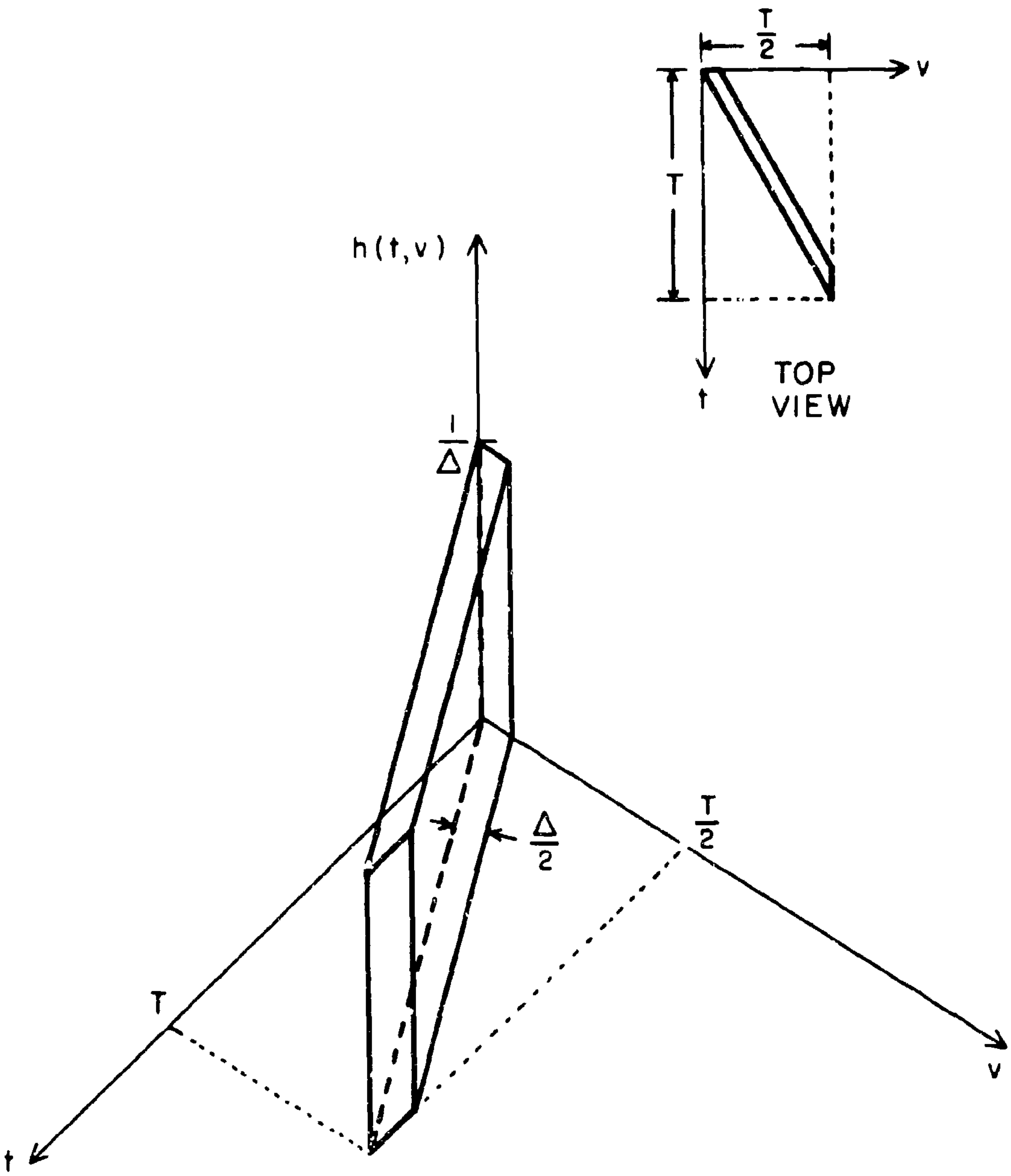


FIGURE 3.5. Plot of h(t, v) on R = (0, T) x (0, T/2) .

and

$$\left\{\psi_k\right\} = \left\{\sqrt{\frac{2}{T}}\, e^{j \frac{4\pi k}{T} t}\right\}_{k=0, \pm 1, \pm 2, \ldots} . \tag{3.40}$$

Then

$$\int_0^{T/2} h(t, v)\, \psi_{-m}(v)\, dv \quad \frac{T\left(1 - e^{-\frac{2\pi m\Delta}{T}}\right)}{j\, 4\pi m\Delta} \; \frac{e^{-\frac{j2\pi mt}{T}}}{\sqrt{T}} \quad , \tag{3.41}$$

and

$$a_{nm} = \int_0^T \int_0^{T/2} \phi_{-n}(t)\, h(t, v)\, \psi_{-m}(v)\, dv\, dt$$

$$= \frac{T\left(1 - e^{-\frac{2\pi m\Delta}{T}}\right)}{j\, 4\pi m\Delta} \; \delta_{m, -n} \quad . \tag{3.42}$$

The expansion of $h(t, v)$ on R is

$$h(t, v) = \sum_{m=-\infty}^{\infty} \frac{T\left(1 - e^{-\frac{2\pi m\Delta}{T}}\right)}{j\, 4\pi m\Delta} \; e^{\frac{j2\pi mt}{T} - \frac{j4\pi mv}{T}} \quad . \tag{3.43}$$

The cross terms are all equal to zero because $e^{j4\pi mv/T}$ and $e^{j2\pi mt/T}$ are shown by Equation (3.41) to be eigenfunctions of the kernel $h(t, v)$.

Taking the transform of Equation (3.43) with respect to v yields

$$H(t, \lambda) = \sum_{m=-\infty}^{\infty} \frac{T \sin \frac{\pi m}{T} \Delta}{2\pi m\Delta} \; \frac{e^{\frac{j\pi m\Delta}{T}}\, e^{\frac{j2\pi mt}{T}}}{\lambda + j\frac{4\pi m}{T}}$$

$$= \sum_{m=-\infty}^{\infty} \frac{T \sin \frac{\pi m}{T} \Delta}{2\pi m\Delta} \left\{ \cos\left(\frac{2\pi mt}{T} + \frac{\pi m}{T}\Delta\right) \frac{2\lambda}{\lambda^2 + \left(\frac{4\pi m}{T}\right)^2} \right.$$

$$\left. + \sin\left(\frac{2\pi mt}{T} + \frac{\pi m}{T}\Delta\right) \frac{\frac{8\pi m}{T}}{\lambda^2 + \left(\frac{4\pi m}{T}\right)^2} \right\} \quad . \tag{3.44}$$

The expansion of $H(t, \lambda)$ in Equation (3. 44) makes evident the realization of $H(t, \lambda)$ as a parallel combination of h-separable networks, as shown in Figure 3. 6, in which a typical branch consists of the network with transfer function

$$\frac{\lambda}{\lambda^2 + \left(\frac{4\pi m}{T}\right)^2}$$

followed by the multiplier

$$\frac{T \sin \frac{\pi m}{T} \Delta}{\pi m \Delta} \cos\left(\frac{2\pi m t}{T} + \frac{\pi m \Delta}{T}\right) ,$$

or the network with transfer function

$$\frac{1}{\lambda^2 + \left(\frac{4\pi m}{T}\right)^2}$$

followed by the multiplier

$$\frac{4 \sin \frac{\pi m}{T} \Delta}{\Delta} \sin\left(\frac{2\pi m t}{T} + \frac{\pi m \Delta}{T}\right) .$$

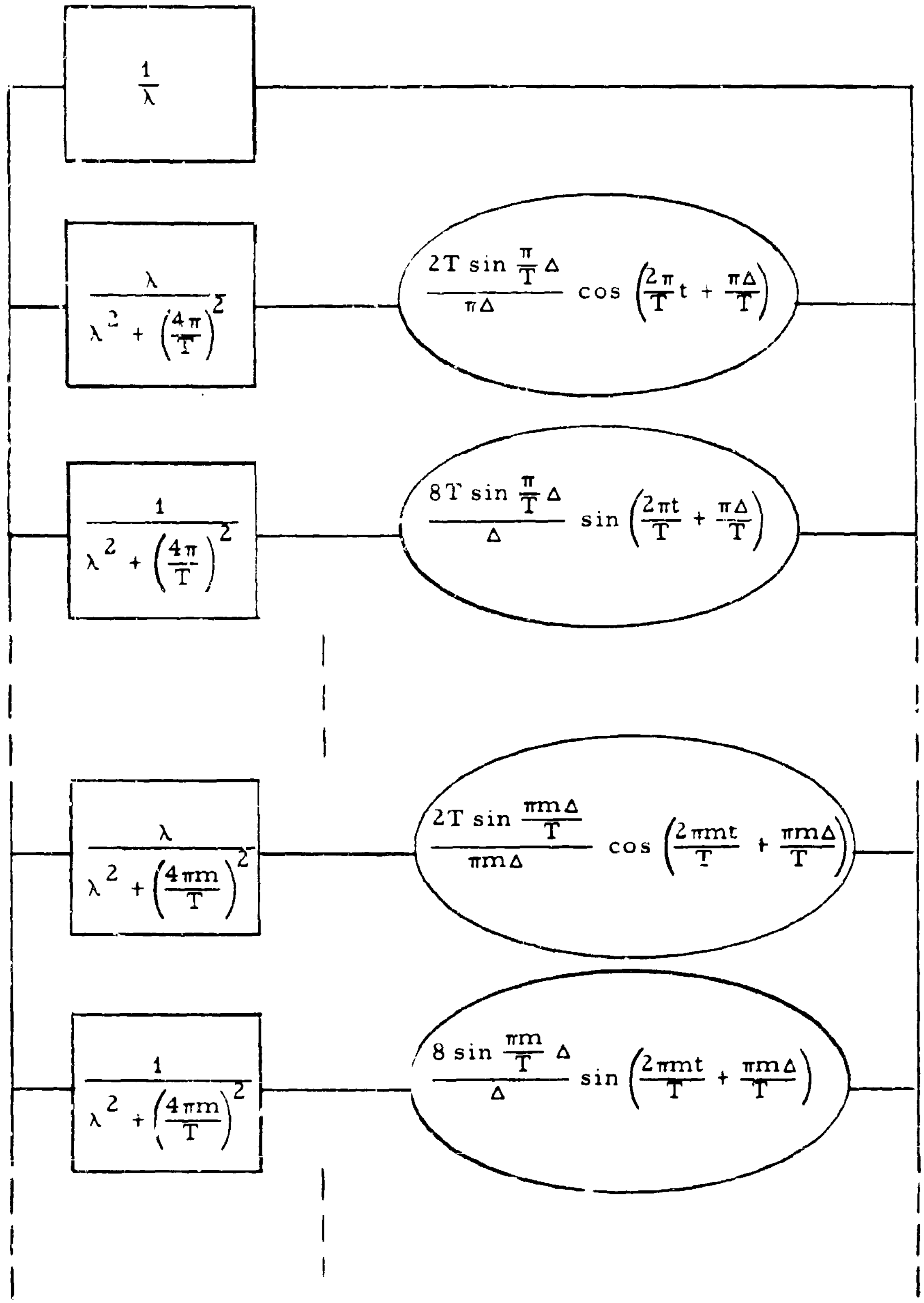


FIGURE 3.6 Realization of h(t, v).

IV. APPLICATIONS OF EXPANSIONS: THE MATRIX APPROACH

A. INTRODUCTION

The expansion of system functions and inputs in appropriate series can lead to the conversion of integral relationships to matrix relationships. The matrix relationships are in some ways easier to handle. The inversion problem was solved by Marcovitz,[9] who used the matrix approach. Matrix methods can be successfully applied to the eigenvalue and eigenvector problem.

B. MATRIX CHARACTERIZATION

Consider the input-output relationship

$$y(t) = \int_c^d k(t, \tau)\, x(\tau)\, d\tau \quad . \tag{4.1}$$

According to Chapter III, one can expand $k(t, \tau)$ in the series

$$k(t, \tau) = \sum_{n,m} a_{nm}\, \phi_n(t)\, \psi_m(\tau) \tag{4.2}$$

on $(a, b) \times (c, d)$. The input is expanded on the interval (c, d) by

$$x(\tau) = \sum_j b_j \psi_j(\tau) \quad . \tag{4.3}$$

The input-output relationship now becomes

$$y(t) = \sum_{n,m,j} a_{nm} b_j \phi_n(t) \int_c^d \psi_m(\tau)\psi_j(\tau)\, d\tau$$

$$= \sum_{n,m} a_{nm} b_m \phi_n(t)$$

$$= \sum_n \left(\sum_m a_{nm} b_m \right) \phi_n(t) \quad . \tag{4.4}$$

The right-hand side of Equation (4.4) is the expansion of the output $y(t)$ on the output time interval (a, b). The coefficient of the expansion of the output is

$$c_n = \sum_m a_{nm} b_m \quad . \tag{4.5}$$

The coefficients of the output expansion are related to the coefficients of the input expansion by the matrix equation

$$[C] = [A]\ B] \quad , \tag{4.6}$$

where

$$[A] = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdot & \cdot & \cdot \\ a_{10} & a_{11} & a_{12} & \cdot & \cdot & \cdot \\ a_{20} & a_{21} & a_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} ,$$

$$B] = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} , \qquad C] = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} . \tag{4.7}$$

One should notice that if the sets $\{\phi_n\}$ and $\{\psi_n\}$ are eigenfunctions of the kernel $k(t,\tau)$, as defined by Equation (? 32), then the expansion of Equation (4. 2) reduces to the expansion of Equation (3. 34). Thus the system matrix A is seen to be diagonal with

$$a_{nm} = \begin{cases} \dfrac{1}{\lambda_m} & \text{for} \quad n = m , \\ 0 & \text{for} \quad n \neq m , \end{cases} \tag{4.8}$$

and the relationship between the input and output coefficients becomes simply

$$c_n = \frac{b_n}{\lambda_n} . \tag{4.9}$$

When the series of Equations (4. 2), (4. 3), and (4. 4) are sampling series of the form of Equations (3. 3) and (3. 5), then for sampling in the quadrant where $t, \tau > 0$,

$$a_{ij} = k_{ij} = k(iT_1, jT_2) . \tag{4.10}$$

By the causality condition,

$$k_{ij} = 0 \quad \text{for} \quad j > i . \tag{4.11}$$

Therefore, the system matrix is triangular

$$[K] = \begin{bmatrix} k_{00} & 0 & 0 & 0 & . & . & . \\ k_{10} & k_{11} & 0 & 0 & . & . & . \\ k_{20} & k_{21} & k_{22} & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} . \tag{4.12}$$

The input-output relationship equation becomes

$$y] = [K] \; x] . \tag{4.13}$$

where

$$y] = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ . \\ . \\ . \end{bmatrix} \quad \text{and} \quad x] = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ . \\ . \\ . \end{bmatrix} \tag{4.14}$$

are the coefficients of the output and the input with samples taken at intervals of T_1 and T_2 sec., respectively.

The j^{th} column of the system matrix $[K]$ has the particular significance of being the output sampled at every iT_1 sec due to a unit impulse input applied at time jT_2. Similarly, the j^{th} column of the system matrix $[A]$ gives the output coefficients due to an input $\psi_j(t)$.

C. THE INVERSION PROBLEM

The inversion problem consists of finding a system which, when placed in cascade with a given system, makes the over-all system unity. It arises in many applications, i.e. signal recovery, feedback systems, and in relating time-varying impedance and time-varying admittance functions.

The two systems in cascade shown in Figure 2.1 have an over-all system function $K_1(s, \tau)$. From the definition of the $K_1(s, \tau)$ system function, a unity system has the response function

$$K_1(s, \tau) = e^{-s\tau} \quad . \tag{4.15}$$

Solving the inversion problem is then equivalent to solving the integral equation of the over-all $K_1(s, \tau)$ function

$$e^{-s\tau} = \int_{-\infty}^{\infty} K_1^{(2)}(s, u)\, k^{(1)}(u, \tau)\, du \tag{4.16}$$

for the left-hand system function $k^{(1)}(t, \tau)$, or the right-hand system function $K_1^{(2)}(s, \tau)$, when the other system function is given. Not only it is difficult, in general, to solve Equation (4.16) for the inverse system function, but it is also difficult to tell when a solution exists at all. For a time-varying delay system, it is possible to find a condition under which an inverse system function exists and to find the inverse system function itself. Let the left-hand system be a delay system with delay $a(\tau)$:

$$k^{(1)}(t, \tau) = \delta(t - \tau - a(\tau)) \quad . \tag{4. 17}$$

Also, let the notion of inversion be extended to allow the over-all system to have a constant delay response rather than a unity response. Then Equation (4. 16) becomes

$$e^{-s(\tau + D)} = \int K_1^{(2)} (s, t)\, \delta(t - \tau - a(\tau))\, dt$$

$$= K_1^{(2)} (s, \tau + a(\tau)) \quad . \tag{4. 18}$$

where the delay constant D is chosen to assure realizability of the inverse system. Thus,

$$k^{(2)}(t, \tau + a(\tau)) = \delta(t - \tau - D) \quad . \tag{4. 19}$$

In order to solve for the right-hand inverse, $\tau + a(\tau)$ must be invertible (i.e. strictly monotonic in τ), in which case

$$k^{(2)} (t, \tau) = \delta(t - x(\tau)) \tag{4. 20}$$

where

$$x(\tau + a(\tau)) = \tau + D \quad . \tag{4. 21}$$

From the form of Equation (4. 21) it is seen that the right-hand inverse is a delay system with delay $x(\tau) - \tau$. For example, if $a(\tau) = \tau$, then $x(\tau) - \tau = \tau/2 + D - \tau = D - \tau/2$. Note that if $D = 0$, delay by τ in the left-hand system is compensated for by prediction of $\tau/2$ in the right-hand inverse system. If the input to the first system is zero for

$\tau > D$, the right-hand inverse is a realizable delay system (i.e. no prediction). Thus D must be chosen large enough, depending on the input, so that the right-hand inverse will be realizable. As another example, delay by $a(\tau) = \tau^3 - \tau$ is compensated for by delay of $D - (\tau - \tau^{1/3}) = D + \tau^{1/3} - \tau$.

For a simple example where the right-hand inverse system function does not exist, take $a(\tau) = D - \tau$. Then $\tau + a(\tau) = D$, which has no inverse. The physical explanation for this is that the first system delays the input $\delta(t - \tau)$ by $D - \tau$, giving an output $\delta(t - D)$ as the input to the second system. The second system sees the same input for every value of τ and obviously has an output that cannot depend on the value of τ; hence the output cannot be $\delta(t - \tau - D)$.

If the right-hand system is given as a delay system with delay $b(t)$, then

$$K_1^{(2)}(s, t) = e^{-s(t + b(t))} \quad . \tag{4.22}$$

Equation (4. 16) now becomes

$$e^{-s(\tau + D)} = \int e^{-s(t + b(t))} k^{(1)}(t, \tau)dt \quad . \tag{4.23}$$

If $t + b(t)$ is invertible, there exists a function $y(t)$ such that $y(t + b(t)) = t$. Now the change of variables $t + b(t) = u$ can be made in Equation (4. 21), giving

$$e^{-s(\tau + D)} = \int e^{-su} k^{(1)} (y(u), \tau) \; y'(u) \, du \quad . \tag{4.24}$$

Taking the transform of Equation (4.24) with respect to s yields

$$\delta(t - \tau - D) = \int \delta(t - u)\, k^{(1)}(y(u), \tau)\, y'(u)\, du$$

$$= y'(t)\, k^{(1)}(y(t), \tau) \qquad ; \qquad (4.25)$$

thus,

$$k^{(1)}(t, \tau) = \frac{1}{y'(t+b(t))} \delta(t + b(t) - \tau - D) \qquad . \qquad (4.26)$$

For example, if the second system has delay $b(t) = t$, then $y(t) = t/2$, and

$$k^{(1)}(t, \tau) = 2\ \delta(2t - \tau - D) \qquad . \qquad (4.27)$$

As another example, let $b(t) = t^3 - t$; then $y(t) = t^{1/3}$, and

$$k^{(1)}(t, \tau) = 3t^2\ \delta(t^3 - \tau - D) \qquad . \qquad (4.28)$$

A simple example of the nonexistence of an inverse is when $b(t) = D - t$, in which case $b(t) + t$ does not have an inverse. It should be noted that by using system-function duality, the discussion here is made equally pertinent to the finding of the inverses of a selective frequency-shift system.

The matrix approach can be used in the inversion problem with a good deal of success. For the system with a matrix A followed in cascade by the system with matrix B, the over-all system matrix is BA. The right hand quasi-inverse to the system K of Equation (4.12)

is defined as the system K' so that

$$K'K = \left[\begin{array}{c|c} 0 & 0 \\ \hline I & 0 \end{array}\right] \quad . \tag{4.29}$$

The system K followed in cascade with the system K' is an identity within a finite amount of delay (finite number of rows of zeros). The matrix K may be finite or infinite.

The necessary and sufficient conditions for the existence of a quasi-inverse are due to Marcovitz.[9] Using the following notation:

K_i is the $i \times i$ submatrix of K containing the first i rows and i columns,

K_{ij} is the submatrix of K_{i+j} containing the first i columns, and

K'_{ij} is the submatrix of K_{i+j} containing the last j columns,

Marcovitz's main theorem states that, if the rank of K_{i+j} is r, the necessary and sufficient conditions for the existence of a quasi-inverse with j units of delay are that the rank of K'_{ij} be $r - i$. The theorem verifies the obvious facts that a sufficient condition for the existence of an inverse is that the terms on the main diagonal are nonzero, i.e.

$$k_{ii} \neq 0 \quad , \tag{4.30}$$

and that the sufficient conditions for the existence of a quasi-inverse with minimum delay j is that the main diagonal and the $j - 1$ diagonals below it be zero and that the j^{th} diagonal below it have all nonzero

terms; i.e.

$$k_{i+l,\,l} = 0 \quad \text{for } i < j, \quad l = 0, 1, 2, \ldots \tag{4.31}$$

$$k_{j+l,\,l} \neq 0 \quad \text{for } l = 0, 1, 2, \ldots \quad . \tag{4.32}$$

The necessary and sufficient conditions for existence of a unique quasi-inverse are also given by Equations (4.31) and (4.32).

An interesting corollary is that, if the first j diagonals are zero as in Equation (4.31) and if $k_{j+l,\,l} \neq 0$ for some value of l, then a necessary condition for the existence of a quasi-inverse is that $k_{j+l,\,l} = 0$, either for an infinite number of l or for no l at all.

Application of the main theorem may be difficult because it may require an infinite number of calculations. The calculations must be applied for each value j of delay until one is found which works. If there is no value of j for which the conditions are satisfied, the quasi-inverse does not exist.

1. Finite Matrices

Finite system matrices occur in coding systems and in the time-limited approximations to the system matrix. For an input that is t units long, one need be concerned with only the first t columns of the system matrix K. The quasi-inverse with respect to an input t units long exists if, and only if, the rank of K_t is t, where K_t is the matrix of the first t columns of K. The maximum delay in recovering the input is given by the number of the last of the t independent

rows of K_t, where the first row is numbered zero. If some of the input digits are zero, then one should consider the submatrix $K_{(r)}$ of K_t, where $K_{(r)}$ is the matrix of the r columns of K which correspond to the r nonzero digits of the input. Since there may be cases where the rank of K_t is not t, but the rank of $K_{(r)}$ is r, the quasi-inverse with respect to a t digit input that has some zeros may exist when the quasi-inverse with respect to a t digit input does not exist.

Let K_{mr} denote the finite submatrix of the infinite matrix K containing the r columns of K that correspond to the r nonzero information bits of the t digit input, and terminating after m rows. If K_{mr} is of rank r, then a quasi-inverse with at most $j=m-r$ units of delay exists. It would be particularly interesting to find the smallest value of m for which the matrix K_{mr} has rank r so that the delay in signal recovery is the shortest. This is done by testing the rank of K_{mr} sequentially in m for $m = r, r+1, r+2, \ldots$ until its rank is found to be r. If there exists no value of m for which the rank of K_{mr} is r, then the quasi-inverse does not exist for that input.

Let K'_{mm} be the quasi-inverse of K_{mr} if it exists, then

$$K'_{mm}K_{mr} = \begin{bmatrix} 0_{jr} \\ \hline I_{rr} \end{bmatrix} \quad . \tag{4.53}$$

The matrix K'_{mm} can be augmented to be a quasi-inverse of the system matrix K with respect to a particular input:

$$K' = \left[K''_{m+z,\, m} \;\middle|\; 0 \right] \quad , \tag{4.34}$$

where

$$z = t - r \qquad , \tag{4.35}$$

and $K''_{m+z,\,m}$ is K'_{mm} with z rows of zeros added to correspond to the z zeros in the input. For example, if $m = 5$ and $z = 3$ and the zeros occurred at the 0^{th}, 3^{rd}, and 5^{th} digits, then the 0^{th}, 3^{rd}, and 5^{th} rows of $K''_{m+3,\,m}$ are zero, and the 1^{st}, 2^{nd}, 4^{th}, 6^{th}, and 7^{th} rows of $K''_{m+3,\,m}$ are the 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, and 5^{th} rows of K'_{mm}, respectively. Enough columns of zeros are added to $K''_{m+z,\,m}$ to make the matrix K' compatible with K_t:

$$K'K_t = \begin{bmatrix} 0_{jt} \\ \hline I'_{tt} \end{bmatrix} \qquad , \tag{4.36}$$

where I'_{tt} has ones on the main diagonal corresponding to the nonzero digits of the input and has zeros elsewhere. From Equation (4.36) one sees that when the input is passed through the system K followed by the system K', the output is j units of zero followed by the input. The recoverability of a t digit input is therefore dependent on the existence of a quasi-inverse to the finite matrix K_{mr}.

2. The Inverse in Feedback Systems

The response to the feedback system of Figure 4.1 can be obtained from the equation

$$\delta y = k_A x + k_B \otimes k_A y \qquad , \tag{4.37}$$

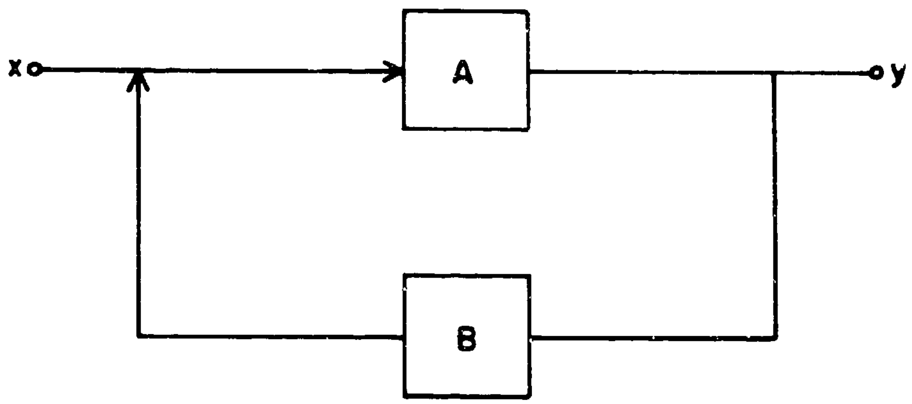


FIGURE 4. 1. Feedback System.

where $k_A x$ means the system function k_A operating on x. Equation (4. 37) can be rewritten as

$$(\delta - k_B \otimes k_A) y = k_A x \qquad . \qquad (4.38)$$

Thus,

$$y = \left[(\delta - k_B \otimes k_A)^{-1} \otimes k_A \right] x = kx \qquad . \qquad (4.39)$$

The inverse to $(\delta - k_B \otimes k_A)$ must be found in order to find the over-all system function k. In matrix notation convolution is replaced by matrix multiplication and the system matrix is given by

$$K = (I - BA)^{-1} A \qquad , \qquad (4.40)$$

where B and A are the system matrices of k_B and k_A, respectively. By using the methods given earlier in this section, one can find the inverse of $I - BA$ if it exists. If a quasi-inverse $(I - BA)'$ exists,

$$(I - BA)'(I - BA) = \begin{array}{c} j \\ \\ \end{array} \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} = D \quad . \tag{4.41}$$

Then

$$DK = (I - BA)' A \quad . \tag{4.42}$$

The first j rows of the right-hand side of Equation (4.42) must be zero.

3. Relationship Between Impedance and Admittance Functions and Equivalence to the Inversion Problem

The K_{1Y} and k_Z functions can be related by inserting Equation (1.26) into Equation (1.27) and interchanging integrals:

$$I(s) = \int K_{1Y}(s, u)\, e(u) du = \int \left[\int K_{1Y}(s, u)\, k_Z(u, \tau) du \right] i(\tau)\, d\tau \quad . \tag{4.43}$$

By the uniqueness of Fourier transforms,

$$e^{-s\tau} = \int K_{1Y}(s, u)\, k_Z(u, \tau) du \quad . \tag{4.44}$$

The following dual relations:

$$e^{-s\tau} = \int K_{1Z}(s, u)\, k_Y(u, \tau)\, du \quad , \tag{4.45}$$

$$e^{\lambda t} = \int K_{2Y}(t, p)\, K_Z(p, \lambda) dp \quad , \tag{4.46}$$

$$e^{\lambda t} = \int K_{2Z}(t, p)\, K_Y(p, \lambda)\, dp \quad , \tag{4.47}$$

are obtained by physical-quantity duality, system-function duality and complete duality, respectively. These relationships can also be written in terms of G and H functions by using identities. By comparison of Equations (4.44) through (4.47), the problem of finding an impedance driving-point system function from an admittance driving-point function, or vice-versa, is seen to be equivalent to finding a left or a right inverse system to a given system.

D. MINIMUM DISTORTION SIGNALS AND SYSTEMS

The problem of finding signals that will pass through a given system without distortion other than a multiplicative constant is an eigenvector problem. From the input-output relationship of Equation (4.13), one sees that the zero-distortion inputs are the eigenvectors of the following characteristic equation:

$$\lambda x] = [K] \; x] \quad . \tag{4.48}$$

Since K is triangular, the eigenvalues are clearly

$$\lambda_j = k_{jj} \quad . \tag{4.49}$$

The eigenvector corresponding to the eigenvalue λ_j is

$$x^{(j)} = \left\{ x_1^{(j)}, \; x_2^{(j)}, \; x_3^{(j)}, \; \ldots \right\} \quad . \tag{4.50}$$

One may assume that the eigenvalues are all nonzero, as the zero eigenvalues are of no interest. The eigenvector $x^{(j_0)}$ can be found

by an iterative method. If λ_{j_0} is not repeated, i.e.

$$k_{j_0 j_0} \neq k_{ll} \quad \text{for} \quad l \neq j_0 \quad , \tag{4.51}$$

then the components of $x^{(j_0)}$ can easily be solved for

$$x_k^{(j_0)} = \begin{cases} 0 \text{ for } k = 1, 2, \ldots, j_0 - 1 \quad , \\ x_{j_0} \text{ for } k = j_0 \quad , \\ \sum_{i=j_0}^{k-1} \frac{a_{ki} x_i^{(j_0)}}{\lambda_{j_0} - \lambda_k} \quad \text{for } k = j_0 + 1, \ldots \; , \end{cases} \tag{4.52}$$

where $x_{j_0}^{(j_0)}$ is an arbitrary constant.

If λ_j is repeated, assume it occurs first at k_{jj}. The matrix A_{jj}, where

$$A_{jj} = \begin{bmatrix} k_{11} & & \boxed{0} \\ \cdot & \cdot & \\ \cdot & & \cdot \\ \cdot & & & \cdot \\ k_{1j} & \cdot & \cdot & \cdot & k_{jj} \end{bmatrix} \quad , \tag{4.53}$$

has the eigenvector $\{0, \ldots, 0, x_j\}$ corresponding to λ_j with x_j chosen arbitrarily. The matrix $A_{j+1, j+1}$ has the eigenvector $\{0, \ldots, 0, x_j, x_{j+1}\}$ corresponding to λ_j, where x_{j+1} is determined by the equation:

$$(\lambda_j - \lambda_{j+1}) x_{j+1} = k_{j, j+1} x_j \quad , \tag{4.54}$$

if $\lambda_j \neq \lambda_{j+1}$. But if $\lambda_j = \lambda_{j+1}$, then either $k_{j,j+1}$ or x_j must be zero. In either case x_{j+1} is arbitrary, but if $k_{j,j+1}$ is nonzero, x_j is no longer arbitrary and must in fact be zero.

Suppose the first repetition of λ_j occurs at λ_{j+r}, then the eigenvector of $A_{j+r,j+r}$ is

$$\left\{\underbrace{0,\ldots,0}_{j},\ x_j, x_{j+1}, \ldots, x_{j+r-1}, x_{j+r}\right\} \quad ,$$

where $x_{j+1}, x_{j+2}, \ldots, x_{j+r-1}$ have been computed by considering the eigenvectors of λ_j in the matrices $A_{j+1,j+1}, A_{j+2,j+2}, \ldots, A_{j+r-1,j+r-1}$, respectively. The new component x_{j+r} is determined by the equation

$$(\lambda_j - \lambda_{j+r})x_{j+r} = \sum_{l=j}^{j+r-1} k_{l,j+r}\, x_l \quad . \tag{4.55}$$

Since the left-hand side of Equation (4.55) is zero, x_{j+r} is arbitrary. But if the right-hand side is not zero, then x_j must be zero, in which case $x_{j+1}, x_{j+2}, \ldots, x_{j+r-1}$ are all zero. Similarly, if the next repetition of λ_j occurs at λ_{j+r+t} then

$$\sum_{l=j}^{j+r+t-1} k_{l,j+r+t}\, x_l$$

must be zero and x_{j+r+t} is arbitrary. Since the right-hand side of Equation (4.55) is zero, this is equivalent to saying that

$$\sum_{l=j+r}^{j+r+t-1} k_{l,\,j+r+t}\, x_l$$

must be zero. If it is nonzero, then x_{j+r} must be zero, in which case $x_{j+r+1}, \ldots, x_{j+r+t-1}$ are all zero. This process is now continued, and it is clear what happens at the next repetition of λ_j.

It may be possible that all eigenvectors of a given infinite matrix may be zero, as in the matrix

$$K = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & \lfloor 0 & \\ & 1 & 1 & & \\ & & 1 & . & \\ \overline{0}\rceil & & & \cdot & \cdot \\ & & & & \cdot \end{bmatrix} . \qquad (4.56)$$

In finite triangular matrices where eigenvalue λ_j is repeated J times, there is at least one nonzero eigenvector because the component at the place of the last repetition is arbitrary. Also, the number of zero eigenvectors is equal to the number of times during the iterative process that a previously arbitrary component was forced to be zero.

In a system whose matrix has more rows than columns, the output that is desired for minimum distortion is the input followed by zeros:

$$\left.\frac{\lambda x}{0}\right] = \begin{bmatrix} k_{11} & . & \lfloor 0 \\ \vdots & & \cdot \\ k_{1n} & \cdots & k_{nm} \\ \vdots & & \vdots \\ k_{1m} & \cdots & k_{nm} \\ \vdots & \cdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} . \qquad (4.57)$$

Generally one cannot solve Equation (4.57) because there are more equations than unknowns. But if the row vectors $k_{1m}, \ldots, k_{nm}$ for $m > n$ are orthogonal to a particular eigenvector $x^{(j)}$ of the matrix A, the equation is satisfied where

$$A = \begin{bmatrix} k_{11} & & \lfloor 0 & \\ \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ k_{1n} & \cdot & \cdot & \cdot \; k_{nn} \end{bmatrix} \quad . \tag{4.58}$$

Note that if $x^{(j)}$ is the eigenvector of matrix A of $\lambda_j = k_{jj}$, and $y^{(l)}$ is the eigenvector of its transpose conjugate matrix A^* corresponding to $\lambda_l = k_{ll}$, and if $\lambda_j \neq \lambda_l$, then $x^{(j)}$ and $y^{(l)}$ are orthogonal:

$$\overline{\lambda_l}\left(x^{(j)}, y^{(l)}\right) = \left(x^{(j)}, A^* y^{(l)}\right) = \left(Ax^{(j)}, y^{(l)}\right) = \lambda_j\left(x^{(j)}, y^{(l)}\right) \quad . \tag{4.59}$$

Since $\overline{\lambda_l} \neq \lambda_j$,

$$\left(x^{(j)}, y^{(l)}\right) = \delta_{jl} \quad . \tag{4.60}$$

If $\lambda_i \neq \lambda_j$ for $i \neq j$, then A is diagonable and

$$T\,A\,T^{-1} = \begin{bmatrix} k_{11} & & & \\ & \cdot & \lfloor 0 & \\ \overline{0}\rceil & & \cdot & \\ & & & \cdot \\ & & & & k_{nn} \end{bmatrix} \quad , \tag{4.61}$$

where the j^{th} column of T is $x^{(j)}$ and the j^{th} row of T^{-1} is $y^{(j)}$.

For this case the infinite matrix

$$K = \left[\begin{array}{c|c} A & 0 \\ \hline W & \begin{smallmatrix} \diagdown & 0 \\ B & \diagdown \end{smallmatrix} \end{array}\right] \quad , \tag{4.62}$$

where the rows of W are $y^{(l)}$ for $l \neq j$, passes the signal $x^{(j)}$ without distortion because $\{x^{(j)}, 0\}$ is an eigenvector of K. Similiarly, if the rows of W are made up of $y^{(k)}$ for $k \neq j_1, j_2, \ldots, j_r$, then the system will pass $x^{(j_1)}, x^{(j_2)}, \ldots, x^{(j_r)}$ without distortion.

E. APPLICATION OF DOUBLE SERIES TO THE EIGENFUNCTION PROBLEM

Matrix methods can be used to solve the eigenfunction equation

$$\lambda\psi = \int_a^b k(t, \tau)\, \psi(\tau)\, d\tau \quad . \tag{4.63}$$

One can expand the unknown eigenfunction in terms of the complete set $\{\phi_1\}$:

$$\psi = \sum_{k=1}^{\infty} a_k \phi_k \quad . \tag{4.64}$$

Taking the terminated expansion and substituting it into the eigenfunction equation yields

$$\lambda \sum_{k=1}^{N} a_k \phi_k(t) = \sum_{k=1}^{N} a_k \int_a^b k(t,\tau)\, \phi_k(\tau)\, d\tau \quad . \tag{4.65}$$

Now multiplying both sides by $\phi_j(t)$ and integrating gives

$$\lambda a_j = \sum_{k=1}^{N} a_k \int_a^b \int_a^b \phi_j(t)\, k(t,\tau)\, \phi_k(\tau)\, dt\, d\tau$$

$$= \sum_{k=1}^{N} a_{jk}\alpha_k \quad \text{for } j=1,\ldots,N \quad , \tag{4.66}$$

where a_{jk} is the $(jk)^{th}$ coefficient of the expansion of $k(t, \tau)$:

$$k(t, \tau) = \sum_{j,k=1} a_{jk}\, \phi_j(t)\, \phi_k(\tau) \quad . \tag{4.67}$$

Thus, the integral eigenfunction equation has been reduced to the matrix eigenvector equation

$$\lambda\alpha] = [A]\ \alpha] \quad , \tag{4.68}$$

where

$$\alpha] = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ N \end{bmatrix} \qquad [A] = \{a_{jk}\} \quad .$$

An approximation can be made to the eigenvalue λ and to the coefficients of the expansion of the eigenfunction by solving the matrix characteristic Equation (4.68).

Note that $\{\phi_i\}$ need not be complete but only good enough to approximate $k(t,\tau)$ and ψ to within a olerable error. When the $\{\phi_i\}$ are approximating rectangles, this method reduces to the Fredholm method where one approximates the integral by a sum to reduce the integral equation to a matrix equation. One advantage of this method is that a judicious choice of the set $\{\phi_i\}$ may lead to a better approximate solution for a given N, or a smaller N for a fixed amount of error.

1. Error Evaluation

By the completeness of the set $\{\phi_i\}$ and by Bessel's inequality one can choose N so large that

$$\|\psi\|^2 - \sum_1^N a_j^2 < \epsilon \qquad , \tag{4.69}$$

$$\|k(t,\tau)\|^2 - \sum_{j,k=1}^N a_{jk}^2 < \epsilon \qquad . \tag{4.70}$$

The equation

$$\lambda a_j = \sum_1^\infty a_{jk} a_k \quad \text{for } j = 1, \ldots, N \tag{4.71}$$

gives the exact solution:

$$\psi = \sum_1^\infty a_j \phi_j \qquad ; \tag{4.72}$$

and the equation

$$\lambda' \beta_j = \sum_1^N a_{jk} \beta_k \tag{4.73}$$

gives the approximate solution:

$$\psi' = \sum_1^N \beta_k \phi_k \qquad . \tag{4.74}$$

By Schwartz's inequality,

$$\sum_{N+1}^{\infty} a_{jk}\,a_k < \left(\sum_{N+1}^{\infty} a_{jk}^2\right)^{\frac{1}{2}} \left(\sum_{N+1} a_k^2\right)^{\frac{1}{2}} < \epsilon \quad , \qquad (4.75)$$

and Equation (4.71) can be rewritten

$$\lambda\, a_j = \sum_{1}^{N} a_{jk}\,a_k + O(\epsilon) \qquad . \qquad (4.76)$$

As N becomes increasingly large, Equations (4.71) and (4.74) become virtually identical.

V. SYNTHESIS OF TIME-VARYING SYSTEMS

A. INTRODUCTION

The realization of the double expansion of $h(t, v)$, as demonstrated in the example of Section III D, is applicable to the synthesis problem. The function

$$h(t, v) = \sum_{n,m} a_{nm} \phi_n(t) \psi_m(v) \tag{5.1}$$

on $R = (a, b) \times (c, d)$, where $\psi_m(\tau)$ is a realizable impulse response, was shown to be realizable as a parallel combination of H-separable branches with a typical branch consisting of the network of impulse responses $\psi_m(\tau)$ followed by the multiplier $a_{nm} \phi_n(\tau)$.

The realization by double expansion has certain practical advantages over the method proposed by Cruz and Van Valkenburg[10] in which $h(t, v)$ was realized from the expansion in terms of the complete set $\{\psi_i\}$:

$$h(t, v) = \sum_{m} a_m(t) \psi_m(v) \quad , \tag{5.2}$$

where

$$a_m(t) = \int_c^d h(t, v) \psi_m(v) \, dv \quad . \tag{5.3}$$

The multipliers $\alpha_m(t)$ may be of complicated form and therefore difficult to build, whereas one has some choice over the sets $\{\phi_n\}$. For example, if the interval (a, b) is finite one can choose $\{\phi_n\}$ to be sines and cosines, in which case the multipliers are standard modulating networks. Actually if one were to arrange Equation (5.1) as

$$h(t, v) = \sum_m \left\{ \sum_r a_{nm} \phi_n(t) \right\} \psi_m(v) \quad , \tag{5.4}$$

then by comparison with Equation (5.2),

$$\alpha_m(t) = \sum_n a_{nm} \phi_n(t) \quad . \tag{5.5}$$

Equation (5.5) is seen to be a realization of the multiplier $\alpha_m(t)$ in terms of the elementary multipliers $\phi_n(t)$.

Conversely, consider the single expansion of $h(t, v)$ in terms of a desired set of multipliers $\{\phi_n(t)\}$:

$$h(t, v) = \sum_n \phi_n(t) \beta_n(v) \quad , \tag{5.6}$$

where

$$\beta_n(v) = \int_a^b h(t, v) \phi_n(t) \, dt \quad . \tag{5.7}$$

The network of impulse response $\beta_n(v)$ is not necessarily realizable. The rearrangement of Equation (5.1),

$$h(t, v) = \sum_{n} \left\{ \sum_{m} a_{nm} \psi_m(v) \right\} \phi_n(t) \quad , \tag{5.8}$$

by comparison with Equation (5.6) gives

$$\beta_n(v) = \sum_{m} a_{nm} \psi_m(v) \quad . \tag{5.9}$$

The truncated expansion

$$\beta_n(v) \approx \sum_{m=1}^{M} a_{nm} \psi_m(v) \tag{5.10}$$

gives an approximate realization of $\beta_n(v)$ in terms of the realizable networks $\psi_m(v)$. This approximation is arbitrarily close in the mean square sense for M sufficiently large.

B. THE CROSS-COUPLED REALIZATION*

The H-separable realization of the terminated single expansion,

$$h(t, v) \approx \sum_{m=1}^{M} \alpha_m(t) \psi_m(v) \quad , \tag{5.11}$$

consists of M branches, each containing one passive network followed by one multiplier, whereas the H-separable realization of the terminated

*Suggested by Prof. N. DeClaris

double expansion

$$h(t, v) = \sum_{m, n=1}^{M, N} a_{nm} \phi_n(t) \psi_m(v) \tag{5. 12}$$

consists of MN branches, each containing one passive network followed by one multiplier. This represents an increase on the order of N^2 for the double expansion. There is, however, a great deal of redundancy in the H-separable realization of the double expansion which can be eliminated by the cross-coupled realization, as shown in Figure 5. 1. The cross-coupled network contains M networks and N multipliers with a coupling array linking the networks and multipliers. The number of networks and multipliers is of the same order as in the realization of single expansion.

The coupling array may be realized in a number of ways. Since Figure 5. 1 is actually a flow diagram, there must be a summer at the input to each multiplier. After scaling the largest $|a_{nm}|$ to unity, the potentiometers at the inputs to the summer of multiplier ϕ_i are set to $|a_{ik}|_{k=1, 2, \ldots, M}$, but those a_{ik} that are negative in sign are first passed through an inverter. The coupling network may be a one-element kind of network (resistive for low frequencies or capacitive for high frequencies). This network must have M inputs and either N outputs of the form

$$\sum_{j=1}^{M} a_{ij} \psi_j \qquad ,$$

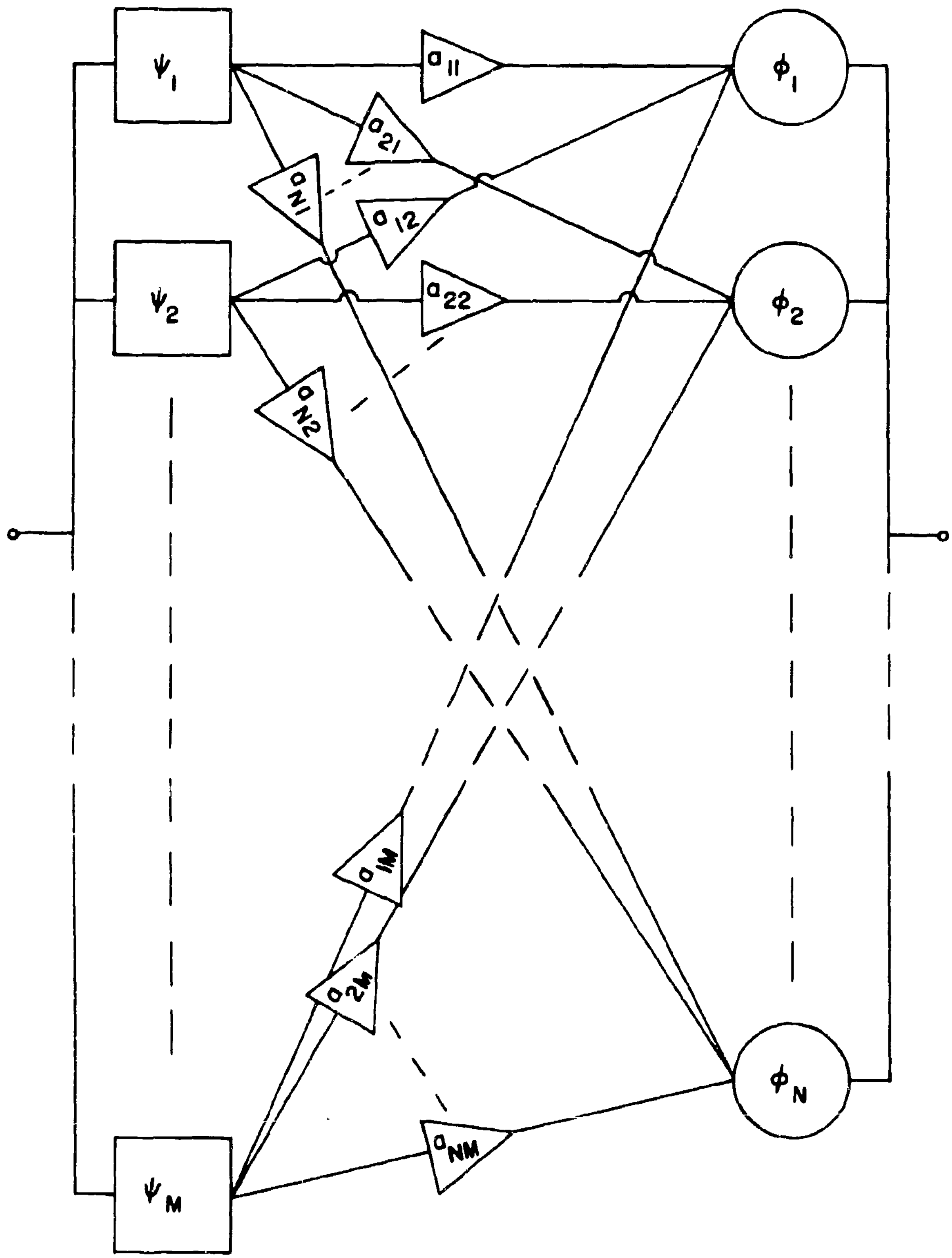


FIGURE 5.1. Cross-Coupled Realization of $h(t, v)$.

or NM outputs of the form $a_{nm}\psi_m$. The latter is obtainable by tapping unity resistors of input voltage ψ_m at values $|a_{nm}|$ and inverting the coefficients of negative sign. Summers must still be built at the inputs to the multipliers, whereas summation is done by the networks in the former case, but the class of coefficients $\{a_{ij}\}$ for which this network can be synthesized is limited by the realizability conditions on the network.

C. BILINEAR FORM

It is useful to think of the realization of the truncated expansion of Equation (5.12) as the realization of its corresponding bilinear form:

$$h(t,v) = \underline{\psi}_M \, A^t_{NM} \, \phi]_N \quad , \tag{5.13}$$

where

$$\underline{\psi}_M = (\psi_1, \psi_2, \ldots, \psi_M) \quad ,$$

$$A_{NM} = \{a_{nm}\} \quad ,$$

$$\phi]_N = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} \quad . \tag{5.14}$$

One can identify the form of Equation (5.13) with the structure of the realization of Figure 5.1 with the cross-coupling network corresponding to the matrix $A^t_{NM} = A_{MN}$.

Suppose that the set $\{\psi_i\}$ is the orthonormalization of a minimal complete set $\{\theta_i\}$ and the set $\{\phi_i\}$ is the orthonormalization of a minimal complete set $\{\gamma_i\}$, then $\underline{\psi}_M$ and $\phi]_N$ are a linear combination of the first M θ_i's and the first N γ_i's, respectively:

$$\underline{\psi}_M = \underline{\theta}_M B_{MM} \quad ,$$

$$\phi]_N = C_{NN} \gamma]_N \quad . \tag{5.15}$$

Since the sets are minimal complete, the matrices B_{MM} and C_{NN} are invertible, and

$$h(t, v) = \underline{\theta}_M B_{MM} A_{MN} C_{NN} \gamma]_N \quad . \tag{5.16}$$

The new cross-coupling matrix is

$$A'_{MN} = B_{MM} A_{MN} C_{NN} \quad . \tag{5.17}$$

In particular, suppose the $\{\psi_i\}$ are the orthonormalization of a set of exponentials,

$$\{\theta_i\} = \left\{e^{s_i t}\right\} \quad . \tag{5.18}$$

One can easily synthesize a ladder network whose poles are $s_1, \ldots, s_M$. The outputs of M rungs of the ladder are linear combinations of exponentials of the pole frequencies:

$$\underline{\eta}_M = \underline{e^{st}}_M \, Q_{MM} \quad . \tag{5.19}$$

One can choose M independent rungs of the ladder so that Q_{MM} i invertible. Now,

$$h(t, v) = \eta_{,M} \, Q_{MM}^{-1} \, A'_{MN} \, Y'_{N} \tag{5.20}$$

has tne realization shown in Figure 5.2 with the cross-coupling matrix

$$A''_{MN} = Q_{MM}^{-1} \, A'_{MN} \quad . \tag{5.21}$$

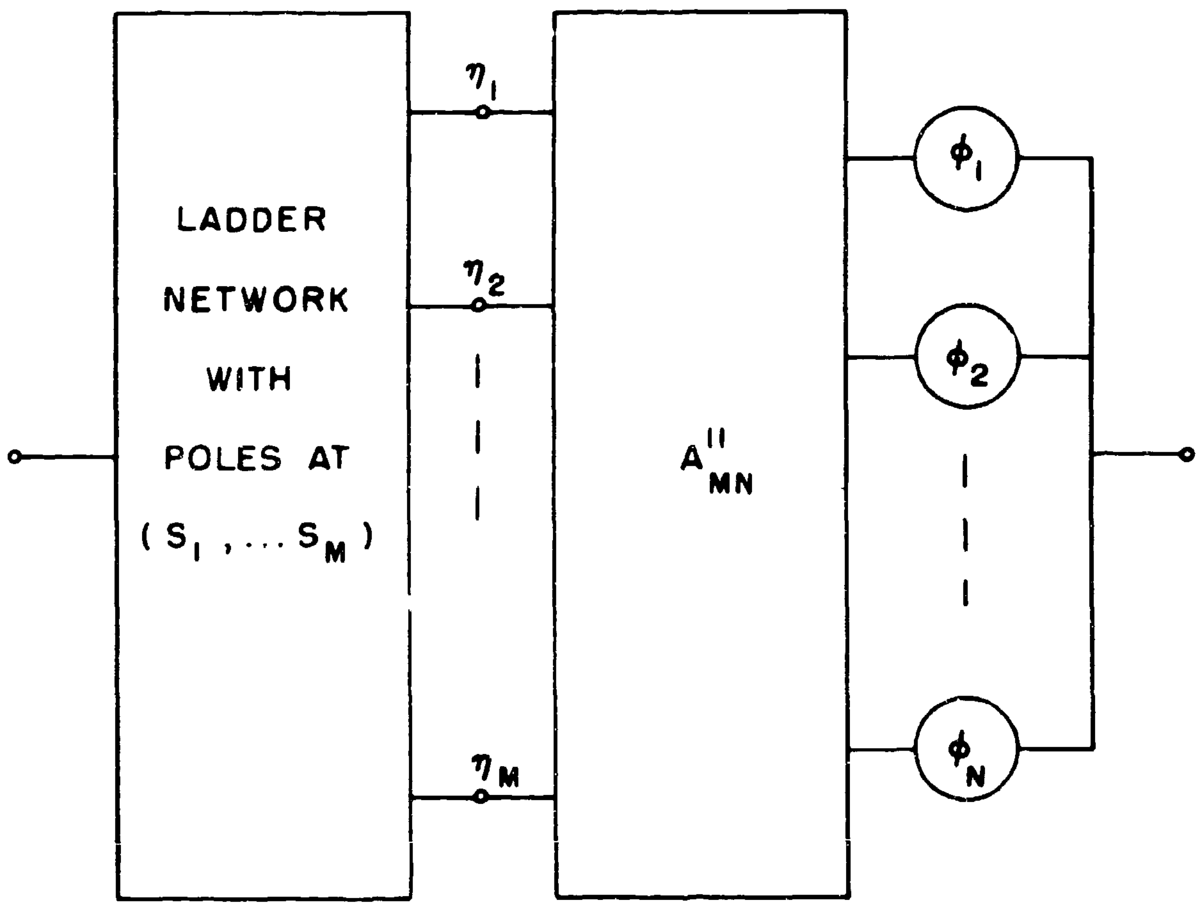


FIGURE 5.2. Realization of h(t, v) with One Passive Network.

For the case where $M = N$, an interesting N-port synthesis problem arises. If one could synthesize a network with poles at $s_1, \ldots, s_M$, with one input port and N output ports, and with the zeros of the output ports chosen in such a way that

$$Q_{MM} = A'_{MM} \quad , \tag{5.22}$$

then

$$A''_{MM} = I \quad , \tag{5.23}$$

and the coupling network consists only of wires connecting ψ_i to ϕ_i.

D. APPROXIMATION IN ONE DIMENSION

The bilinear form approach can be used in the one-dimensional approximation problem when the expansion in one dimension can be put into a bilinear form with both variables the same. If both $\{\psi_i\}$ and $\{\phi_i\}$ are exponential sets, then $h(t, t)$ of Equation (5.20) is expanded in terms of the exponentials $e^{(s_m + s_n)t}$ with coefficients a_{nm}. The realization of $h(t, t)$ is an MN term expansion of $h(t, t)$ obtained by using a network with M poles and N multipliers.

For example, the network of Figure 5.3 is a 16-term realization containing a network whose poles are $s = 0, -2, -4, -6$ and multipliers $1, e^{-8t}, e^{-16t}$, and e^{-24t}, where

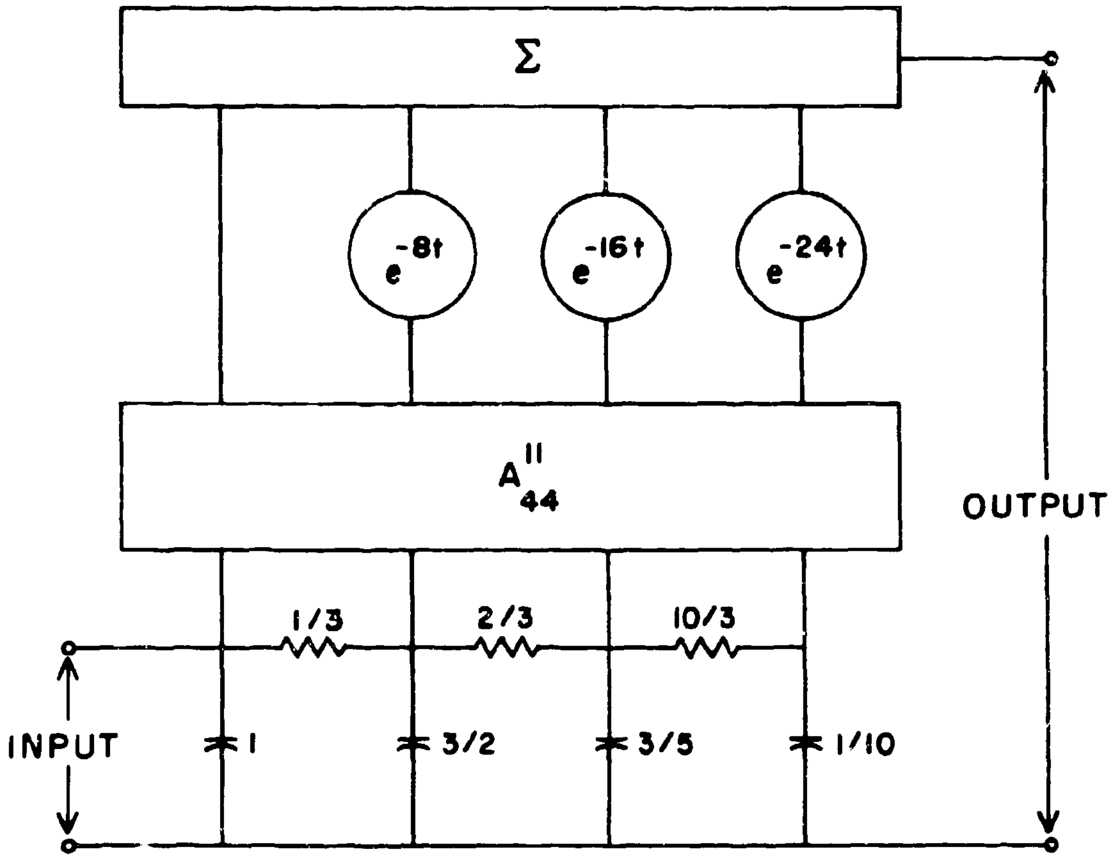


FIGURE 5.3. Sixteen-Term Expansion of h(t, t).

$$
\begin{aligned}
h(t,t) &= \sum_{n,m=1}^{4} a_{nm}\, e^{-(2m+8n)t + 10t} \\
&= \sum_{k=0}^{15} a_k\, e^{-2kt} \qquad\qquad (5.24)
\end{aligned}
$$

E. REALIZATION BASED ON SAMPLING EXPANSION

The realization of the sampling series

$$k(t, \tau) = \sum_{j=0}^{\infty} k\left(\frac{j}{2B}, \tau\right) \text{ sinc } 2\pi B \left(t - \frac{j}{2B}\right) \tag{5.25}$$

using a delay line and a summer, as shown in Figure 5.4, was proposed by Kailath.[4] A similar realization, based on the link structure, is shown in Figure 5.5. The advantage of the link structure is that the summation is performed by the plate delay line.

The impulse response of the link structure for $\Delta = 1/2B$, is

$$k(t, \tau) = \sum_{j=0}^{h} a_j\left(\tau + (j+1)\frac{\Delta}{2}\right) \text{ sinc } 2\pi B\left(t - \frac{(j+1)}{2B}\right) \quad . \tag{5.26}$$

The expansions of Equations (5.25) and (5.26) are identical when

$$a_j\left(\tau + (j+1)\frac{\Delta}{2}\right) = k\left(\frac{j}{2B}, \tau\right) \quad . \tag{5.27}$$

Since

$$k\left(\frac{j}{2B}, \tau\right) = \sum_{l=0}^{\infty} k\left(\frac{j}{2B}, \frac{l}{2W}\right) \text{ sinc } 2\pi W\left(t - \frac{l}{2W}\right)$$

$$= \sum_{l=0}^{\infty} a_j\left(\frac{l}{2W} + (j+1)\frac{\Delta}{2}\right) \text{ sinc } 2\pi W\left(t - \frac{l}{2W}\right) \quad , \tag{5.28}$$

the samples of the double expansion of $k(t, \tau)$ are related to the multiplier samples by

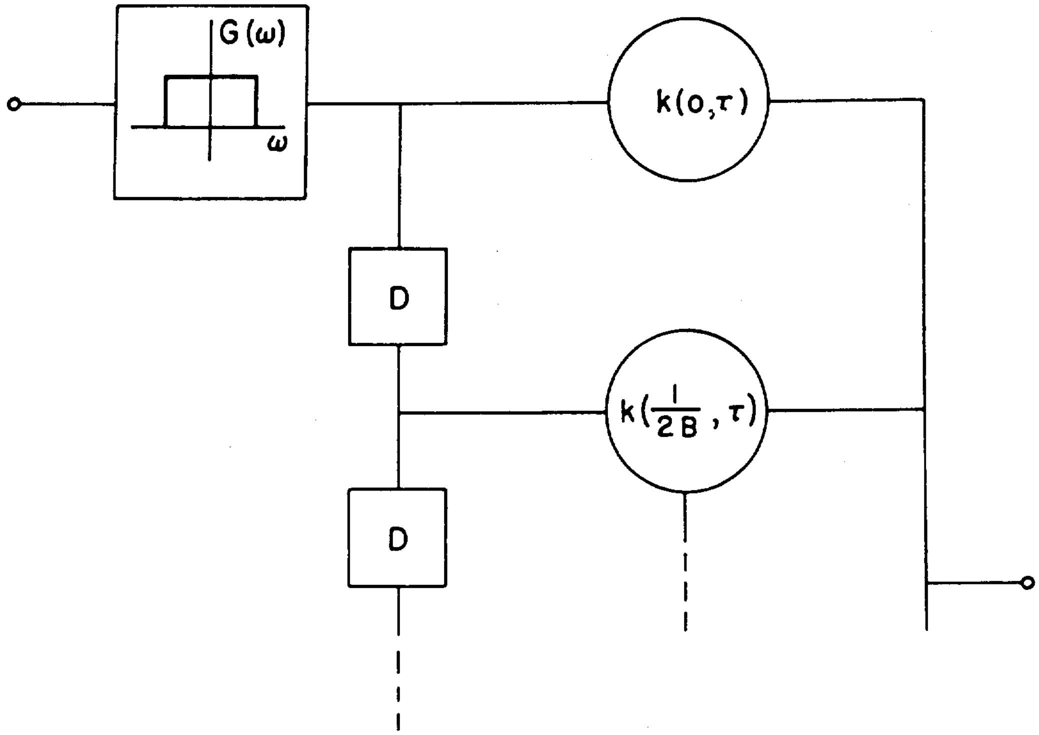


FIGURE 5.4. Realization of $k(t, \tau)$ Based on the Sampling Expansion.

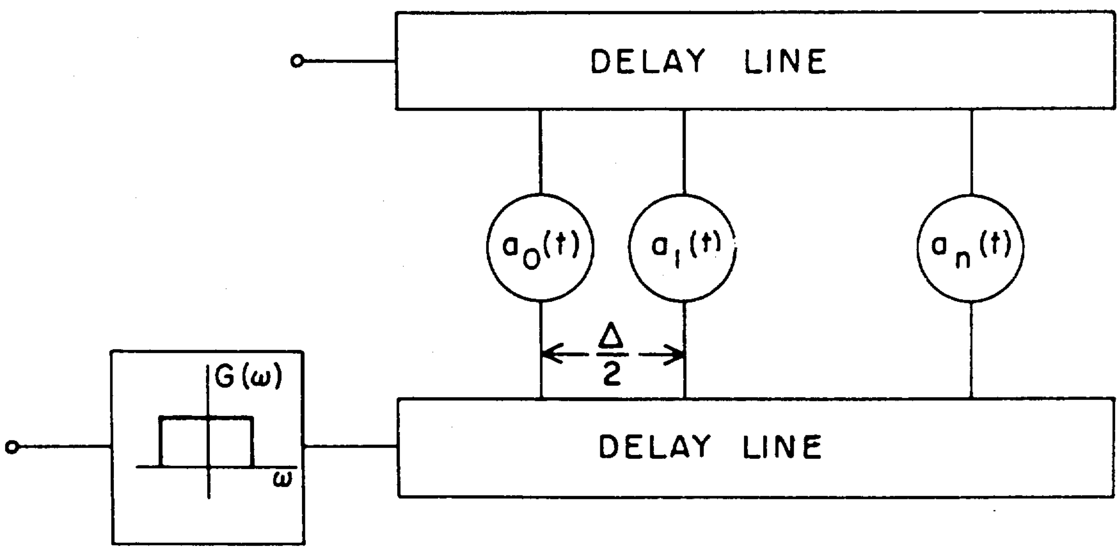


FIGURE 5.5. Sampling Expansion Realization of $k(t, \tau)$ Using Link Structure.

$$k\left(\frac{j}{2B}, \frac{l}{2W}\right) = a_j\left(\frac{l}{2W} + (j+1)\frac{\Delta}{2}\right) \quad . \tag{5.29}$$

Thus the samples of the output due to an impulse at $t = l/2W$ are

$$\left\{k\left(0, \frac{l}{2W}\right), k\left(\frac{1}{2B}, \frac{l}{2W}\right), k\left(\frac{2}{2B}, \frac{l}{2W}\right), \ldots\right\}$$
$$= \left\{a_0\left(\frac{l}{2W} + \frac{\Delta}{2}\right), a_1\left(\frac{l}{2W} + \frac{2\Delta}{2}\right), a_2\left(\frac{l}{2W} + \frac{3\Delta}{2}\right) \ \ldots\right\} \quad . \tag{5.30}$$

The multipliers can be set according to Equation (5.27) or for $\Delta = 1/W$ they can be set sequentially; i.e. at $t = \Delta/2$ the multipliers are

$$\left\{a_0\left(\frac{\Delta}{2}\right), 0, 0, \ldots, 0\right\} \quad ,$$

at $t = \Delta$ the multipliers are

$$\left\{a_0(\Delta), a_1\left(\frac{3\Delta}{2}\right), 0, 0, \ldots, 0\right\} \quad ,$$

at $t = 3\Delta/2$ the multipliers are

$$\left\{a_0\left(\frac{3\Delta}{2}\right), a_1(2\Delta), a_2\left(\frac{5\Delta}{2}\right), 0, \ldots, 0\right\} \quad ,$$

etc.

VI. APPLICATIONS AND CONCLUSIONS

Several examples of time-varying systems will be briefly outlined in this chapter. The emphasis of the presentation is on the application of the methods that were presented in previous chapters to these examples.

A. SATELLITE COMMUNICATION SYSTEM

The communication channel between two ground stations via an earth satellite is time-varying because the time delay and Doppler shift that the receiver sees change in time due to the movement of the satellite. If the satellite is active its transfer function may be time-varying due to satellite precession, position of solar cells, etc. For geometric simplicity, assume that the satellite is traveling in a circular orbit of radius R, as shown in Figure 6.1, and that the transmitter, receiver, and satellite are located as indicated at angles α, β and θ, respectively. Some of the notation concerned with this channel follows:

ω_R = angular velocity of satellite,

θ = $-\omega_R t$,

T = $2\pi/\omega_R$ = time for one orbit,

v = $R\omega_R$ = tangential velocity of satellite,

v_c = speed of light,

r = radius of earth,

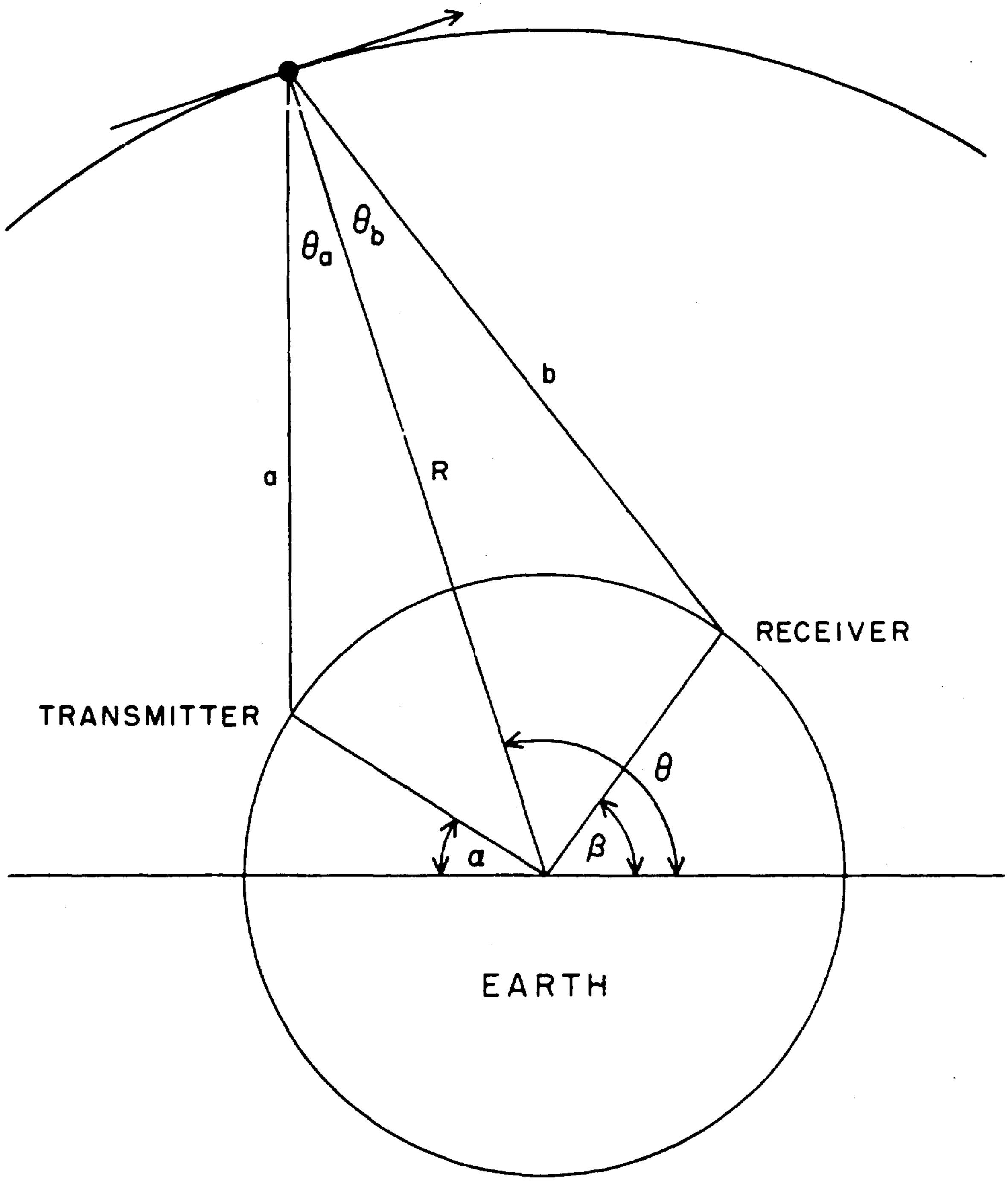


FIGURE 6.1. Geometry of Satellite Communication System.

R = radius of orbit,

Δ_a = time delay between transmitter and satellite,

W_a = Doppler shift between transmitter and satellite,

ω = carrier frequency.

The total amount of delay between the transmitter and receiver is

$$\Delta = \Delta_a + \Delta_b = \frac{a+b}{v_c} \quad , \tag{6.1}$$

where a and b are the transmitter-to-satellite distance and satellite-to-receiver distance, respectively. These distances can be computed by the rule of cosines:

$$a = r^2 + R^2 + 2r R \cos(\theta+\alpha)^{\frac{1}{2}} \quad ,$$

$$b = r^2 + R^2 + 2r R \cos(\theta-\beta)^{\frac{1}{2}} \quad . \tag{6.2}$$

The Doppler shift is proportional to the component of the velocity towards the transmitter or towards the receiver, and for the channel of Figure (6.2),[11] a transmitted carrier of frequency ω is received as a sinusoid of frequency $\omega + W$, where the Doppler shift W is

$$W = (-v n_T + v n_R) \frac{\omega}{v_c} \quad . \tag{6.3}$$

The total Doppler shift in the communication channel is

$$\begin{aligned} W &= W_a + W_b \\ &= \left(-\cos(90 - \theta_a) + \cos(90 - \theta_b)\right) \frac{R\omega_R \omega}{v_c} \end{aligned}$$

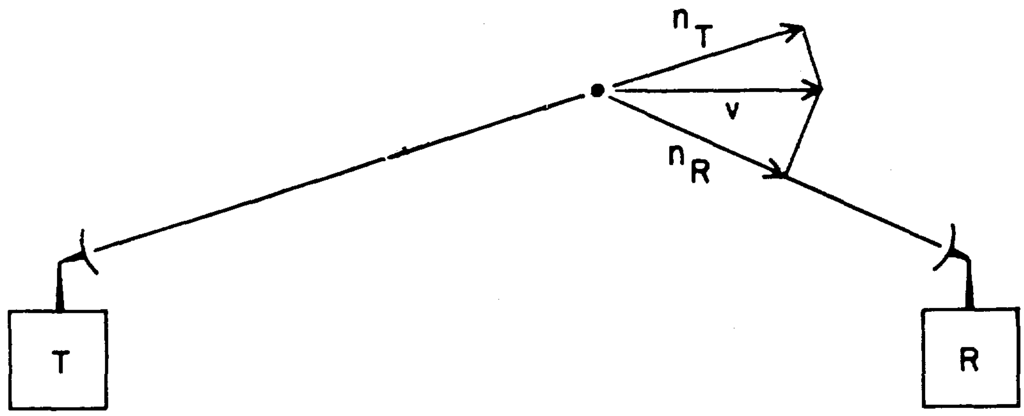


FIGURE 6.2. Doppler Shift Geometry.

$$= (\sin\theta_b - \sin\theta_a)\,\frac{R\omega_R\omega}{v_c} \quad . \tag{6.4}$$

By a trigonometric identity,

$$\frac{r}{\sin\theta_a} = \frac{a}{\sin(180-\theta-\alpha)} = \frac{a}{\sin(\theta+\alpha)} \quad , \tag{6.5}$$

and

$$\frac{r}{\sin\theta_b} = \frac{b}{\sin(\theta-\beta)} \quad . \tag{6.6}$$

Thus,

$$\theta_a = \sin^{-1}\left\{\frac{r}{a}\sin(\theta+\alpha)\right\} \quad ,$$

$$\theta_b = \sin^{-1}\left\{\frac{r}{b}\sin(\theta-\beta)\right\} \quad , \tag{6.7}$$

and the total Doppler shift becomes

$$W = \frac{rR\omega_R\omega}{v_c} \left[\frac{1}{b}\sin(\theta-\beta) - \frac{1}{a}\sin(\theta+\alpha)\right] \quad . \tag{6.8}$$

Now the $K_2(t,\omega)$ system function can be calculated. For a transmitted carrier $e^{j\omega t}$, the satellite receives $e^{j(\omega+W_a)(t-\Delta_a)}$. If the satellite H function is $H(t,\omega)$, the satellite output is then

$$H(t, \omega + W_a) e^{j(\omega+W_a)(t-\Delta_a)} \quad .$$

The received signal becomes $H(t-\Delta_b, \omega+W_a)\, e^{j(\omega+W)(t-\Delta)}$, which for $v \ll v_c$ is approximately $H(t-\Delta_b, \omega+W_a)\, e^{j(\omega+W)t-j\omega\Delta}$. Considering the fact that the satellite first comes into the line of sight of the receiver at time t_β and first goes out of the line of sight of the transmitter at time t_α, the system function is

$$K(t,\omega) = \begin{cases} H(t-\Delta_b, \omega+W_a) e^{j(\omega+W)t-j\omega\Delta} & \text{for} \quad t_\beta+nT < t < t_\alpha+nT \quad , \\ 0 & \text{otherwise} \quad , \end{cases} \tag{6.9}$$

where Δ_b, W_a, W, and Δ are functions of time that have been determined by Equations (6.1) through (6.8).

For a passive satellite, the satellite transfer function is a constant that is close to one; for a stably oriented satellite, the satellite transfer function will be a time-invariant repeater $H(\omega)$. In an unstable mode, the axes of the satellite change direction in time, and thus the satellite receiving antenna and transmitting antenna come

into and out of view with the ground transmitter and receiver, respectively, causing a multiplicative effect on the satellite transfer function. Since the effect of the receiving and transmitting antennae must be taken separately, the satellite transfer function is of the form:

$$H(t, \omega) = \alpha(t) \beta(t) H(\omega) \quad , \tag{6.10}$$

where $\alpha(t)$ and $\beta(t)$ may be periodic functions.

Actually, except for possible time variations in the satellite system function, the change in Doppler shift and time delay will usually be small compared with the signal duration, and the over-all system function can be considered as fixed for signals of short duration. Numerical estimates of Doppler and of delay for the typical parameters,

$$\begin{aligned}
v_c &= 3 \times 10^8 \text{ m/sec}, \\
r &= 4000 \text{ mi} = 6.4 \times 10^6 \text{ m} , \\
R &= 16000 \text{ mi} = 25.6 \times 10^6 \text{ m} , \\
T &= 3 \text{ hr} , \\
\omega_R &= 2\pi/10800 \text{ rad/sec} , \\
\alpha &= 60^\circ , \\
\beta &= 0^\circ , \\
\theta &= \cos^{-1} \tfrac{1}{4} \approx 76^\circ ,
\end{aligned}$$

are:

$$\begin{aligned}
W_a &= -14.8 \times 10^{-6} \omega , \\
W &= 10^{-5} \omega , \\
\Delta_b &= .094 \text{ sec} , \\
\Delta &= .185 \text{ sec} .
\end{aligned} \tag{6.11}$$

For a typical carrier frequency of 4×10^9 cps, the total Doppler shift is 40 kc.

B. AIR-TO-GROUND COMMUNICATION SYSTEM

The signal received at a ground station from a plane or a missile passing by at high speed is distorted by time delay and by Doppler shift in frequency. Under the assumption that the plane is traveling at velocity v in a straight line at a minimum distance h from the ground station, the geometry of the air-to-ground communication system is shown in Figure 6.3.

If communication is begun at time t_o when the missile is at a lateral distance r_o from the station, the delay as a function of time is

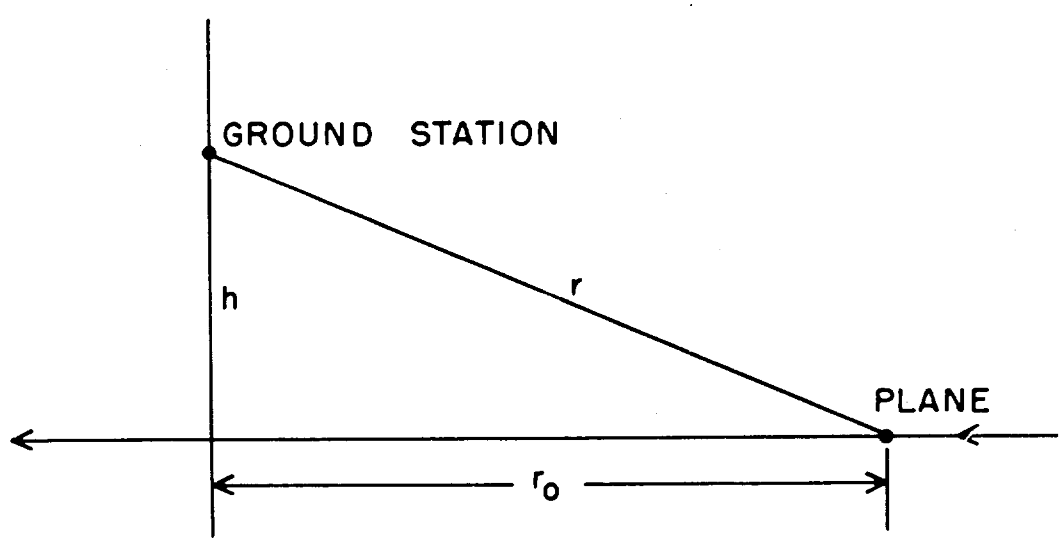


FIGURE 6.3. Geometry of Air-to-Ground Communication System.

$$d(t) = \frac{\sqrt{h^2+(r_o - vt)^2}}{v_c} \tag{6.12}$$

and is plotted in Figure 6.4.

To compensate for the delay $d(t)$, one must construct an inverse filter for it. In Chapter II it was shown that the inverse filter exists if $t - d(t)$ is invertible. Figure 6.5 gives a plot of $t - d(t)$ versus time. The derivative of $t - d(t)$ is

$$\frac{d}{dt}(t - d(t)) = 1 + \frac{v}{v_c} \frac{r_o - vt}{\left[h^2 + (r_o - vt)^2\right]^{\frac{1}{2}}} \quad . \tag{6.13}$$

Since $v/v_c < 1$, the right-hand side of Equation (6.13) is greater than zero for all t, r_o, and h:

$$\frac{d}{dt}(t-d(t)) > 0 \qquad ; \tag{6.14}$$

therefore $t - d(t)$ is a monotonically increasing function and its inverse exists.

The Doppler shift as a function of time is given by

$$W(t) = \frac{-v}{v_c}\omega_o \cos\theta = \frac{-v}{v_c}\omega_o \frac{r_o - vt}{\left[h^2 + (r_o - vt)^2\right]^{\frac{1}{2}}} \quad . \tag{6.15}$$

Figure 6.6 shows this Doppler shift as a function of time. The Doppler shift can be corrected for by using a voltage controlled oscillator in the receiver. If the oscillator control voltage is $W(t)$, the mixer

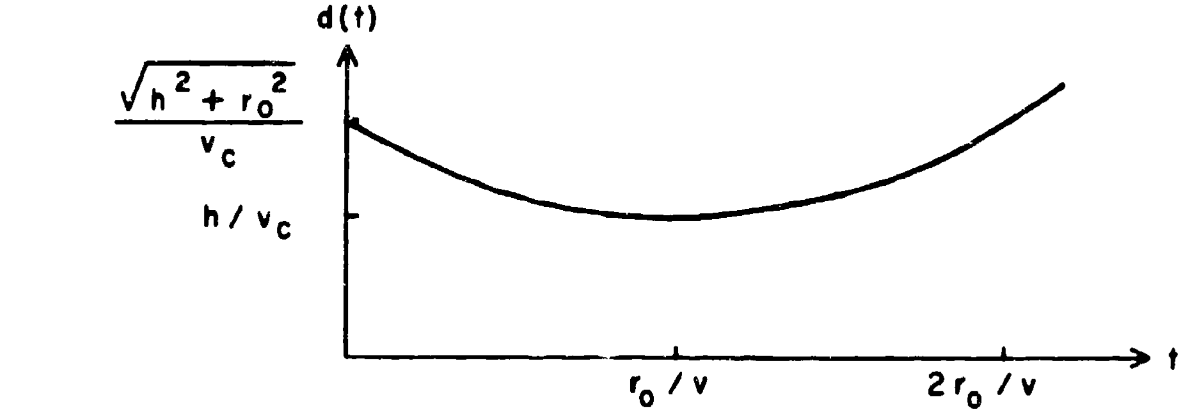


FIGURE 6.4. Delay versus Time.

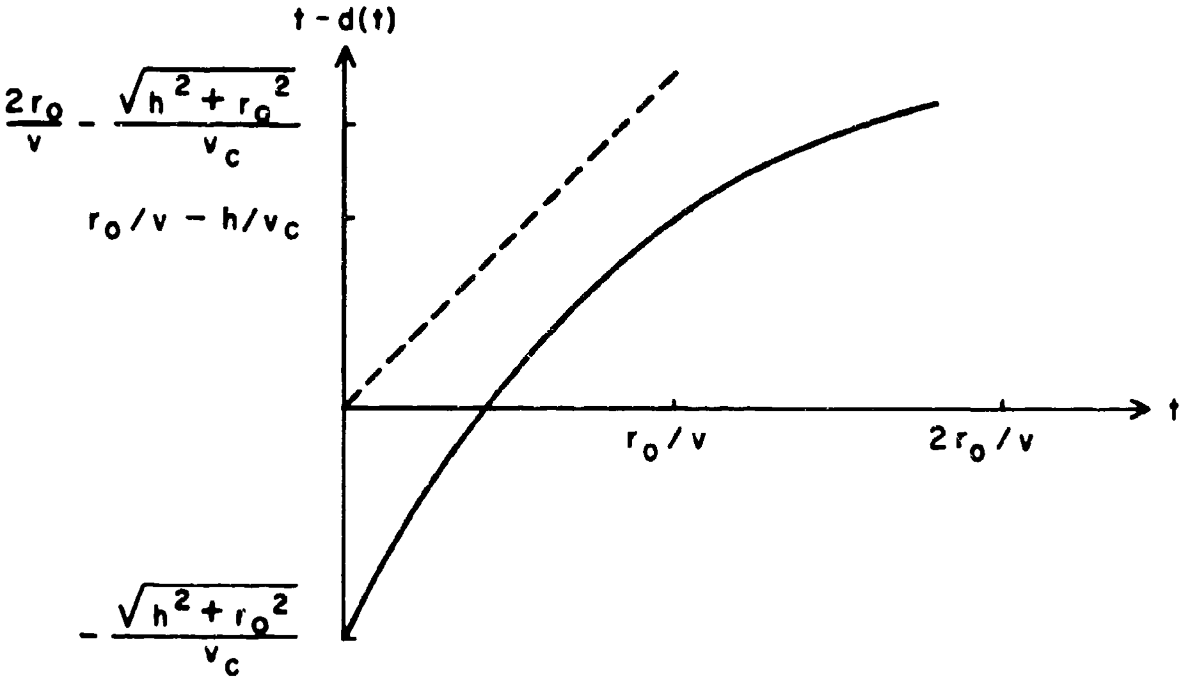


FIGURE 6.5. Plot of $t - d(t)$ versus Time.

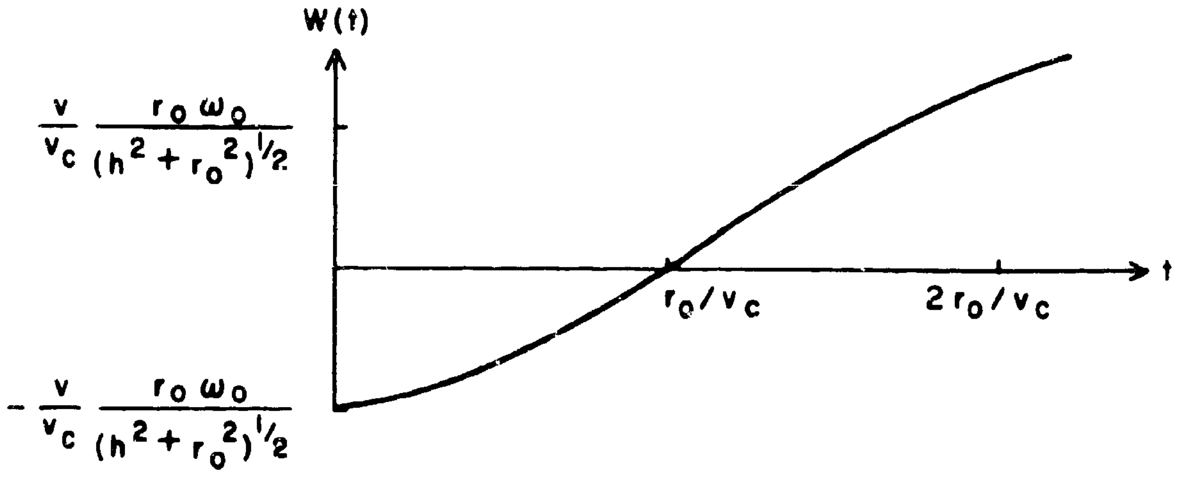


FIGURE 6.6. Doppler Shift versus Time

frequency will be $\omega_1 + W(t)$, and the intermediate frequency will be $\omega_1 - \omega_0$.

C. SIDE-LOOKING RADAR

A plane with a radar unit mounted on its side can be used for an all-weather ground mapping system by flying parallel to the meridian being mapped with the radar antenna facing the target meridian.. The resolution of the mapping is proportional to the carrier frequency of the radar pulses. Ideally the plane will be held on line without turning or rolling, and the time delay and Doppler shift will be constant. Although one can fly a plane parallel to a fixed meridian at a fixed distance, the roll of the plane cannot be controlled perfectly. A typical trajectory of receiving antenna movement due to wing flutter, air drafts, etc., is shown in Figure 6.7.

The time delay will be essentially constant, and since the plane velocity will be small compared with the speed of light and $l' \approx l$, the time delay is approximately

$$\Delta \approx \frac{2l}{v_c} \quad . \tag{6.16}$$

Since the angle α between the incident and reflected waves is small, it can be approximated by

$$\sin \alpha \approx \alpha \approx \frac{\Delta v}{l} \quad , \tag{6.17}$$

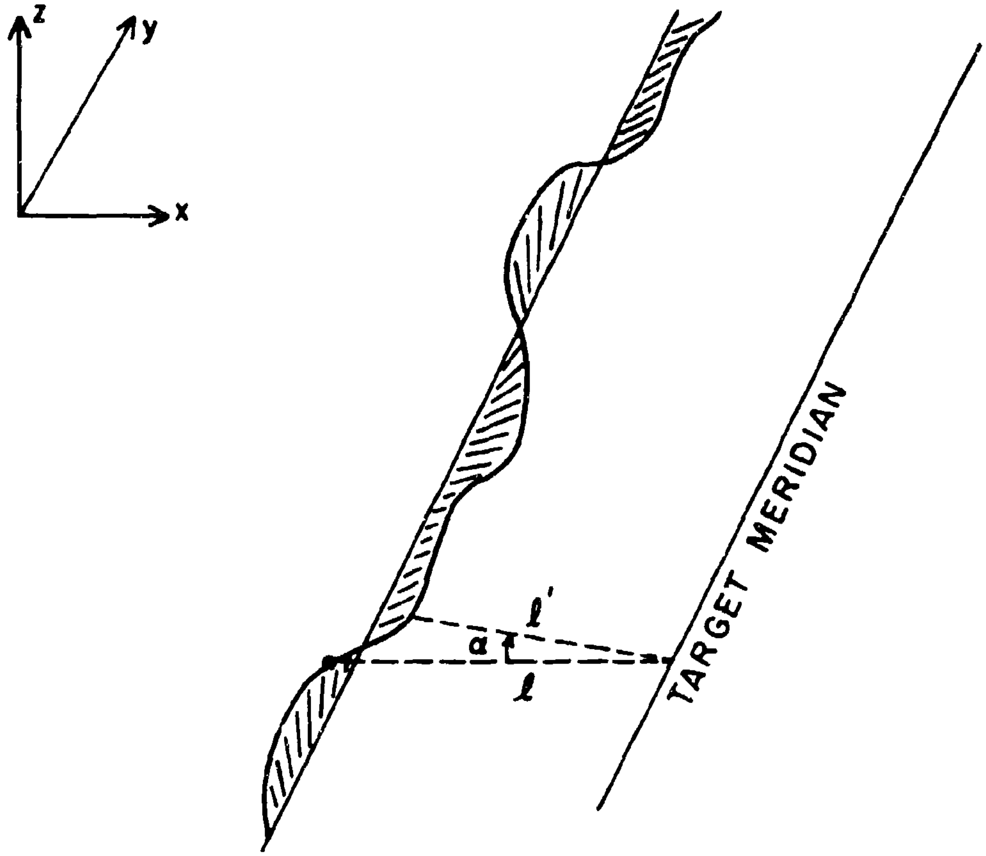


FIGURE 6.7. Side-Looking Radar System.

where v is the plane velocity. During the surveillance flight the received data, as well as the data on plane movement, can be recorded on tape. Afterwards, a curve of the angle between the received wave and the direction of antenna can be computed, and the Doppler shift, which is dependent on this angle, can be compensated for by a voltage-controlled oscillator in the receiver mixer.

D. FEEDBACK CONTROL SYSTEM

Feedback is often used to control the sensitivity of a system to internal variations. If the variations are large the system may be adaptive in that the feedback changes with time to counteract the changing system variations. If the system function for the plant is known, it may be possible to compute the system function of the controller that will give the desired over-all system function. For the system shown in Figure 6.8, the controller A is to be chosen for a given plant B so that the over-all system function will be K.

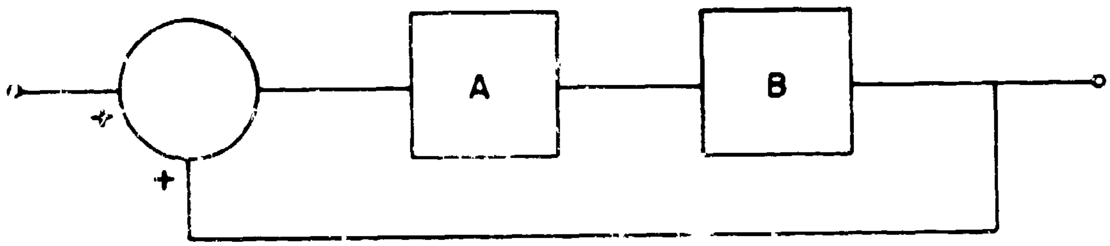


FIGURE 6.8. Feedback Control System.

The matrix of the system function K is

$$K = (I + BA)^{-1} BA$$
$$= I - (I + BA)^{-1} \quad , \qquad (6.18)$$

where B and A are the matrices of systems B and A. Equation (6.18) can be rearranged as

$$BA = (I - K)^{-1} - I \quad , \qquad (6.19)$$

and if B^{-1} exists, then

$$A = B^{-1}(I - K)^{-1} - B^{-1} \quad . \tag{6.20}$$

If only a quasi-inverse of B with j units of delay exists, then

$$DA = B'(I - K)^{-1} - B' \quad , \tag{6.21}$$

where B' is the quasi-inverse of B and

$$D = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right] \begin{array}{c} \uparrow \\ j \\ \downarrow \end{array} = B'B \quad . \tag{6.22}$$

Since the first j rows of the left-hand side of Equation (6.22) are zero, the first j rows of the right-hand side must be zero. This will be true if the first j rows of K are zero. Thus, for a given system B that is quasi-invertible with j units of delay, one can obtain any specified over-all system function K to within j units of delay by choosing the appropriate controlling network A.

As a numerical example, take the plant matrix to be

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad . \tag{6.23}$$

B has the quasi-inverse B' with one unit of delay, where

$$B' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \quad . \tag{6.24}$$

Hence K must be chosen so that the upper left-hand element of $(I - K)^{-1}$ is one:

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix} \quad . \tag{6.25}$$

It can be verified by performing the matrix multiplications indicated by Equation (6.21) that

$$DA = \left[\frac{0}{A}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \end{bmatrix} \quad , \tag{6.26}$$

and therefore the matrix of the controller is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \end{bmatrix} \quad . \tag{6.27}$$

This method is also applicable to infinite matrices.

E. A WHITENING FILTER

A whitening filter for stationary noise $n(t)$ with a nonwhite spectrum $S_n(\omega)$ is a network with transfer function $H(\omega)$ such that

$$\left|H(\omega)\right|^2 = \frac{1}{S_n(\omega)} \quad . \tag{6.28}$$

When $n(t)$ is passed through the network $H(\omega)$, the spectrum of the output noise $n'(t)$ is

$$S_{n'}(\omega) = \left|H(\omega)\right|^2 S_n(\omega) = 1 \quad . \tag{6.29}$$

Thus the output noise $n'(t)$ is white.

If the noise $n(t)$ is nonstationary, then the whitening filter may be time-varying, if it exists. For example, if the noise consists of a Gaussian process with mean zero and spectrum $S_g(\omega)$ plus a time-varying mean $m(t)$:

$$n(t) = n_g(t) + m(t) \quad , \tag{6.30}$$

the whitening filter is a network that subtracts $m(t)$ from the input and then passes $n_g(t)$ through its time-invariant whitening filter. This whitening filter is time-invariant but contains a source (generator of $m(t)$) and a subtractor. If the noise $n(t)$ is of the form

$$n(t) = n_g(t)\, f(t) \quad , \tag{6.31}$$

the whitening filter is a G-separable network with the multiplier $1/f(t)$ followed by the time-invariant whitening filter of $n_g(t)$.

Generally the nonstationary noise will be characterized by its autocorrelation $R_n(t,\tau)$. The whitening filter problem is to find the network $k(t,\tau)$ for a given $R_n(t,\tau)$ such that the output spectrum is $R_{n'}(t-\tau) = \delta(t-\tau)$. Using the noise output

$$n'(t) = \int_0^T k(t,\tau)\, n(\tau)\, d\tau \quad , \tag{6.32}$$

the correlation function of the output noise is given by

$$R_{n'}(t, s) = \int_0^T \int_0^T k(t, \tau)\, k^*(s, u)\, R_n(\tau, u)\, d\tau\, du \quad . \tag{6. 33}$$

The matrix approach can be used by expanding $R_n(\tau, u)$ and the system functions in terms of the complete sets $\langle \phi_i \rangle$ and $\langle \psi_j \rangle$:

$$k(t, \tau) = \sum_{i, j}^{N} a_{ij}\, \phi_i(t)\, \psi_j(\tau) \quad ,$$

$$k^*(s, u) = \sum_{k, l}^{N} a_{kl}^*\, \psi_k^*(s)\, \phi_l^*(u) \quad ,$$

$$R_n(\tau, u) = \sum_{m, p}^{N} r_{mp}\, \phi_m(u)\, \psi_p^*(\tau) \quad . \tag{6. 34}$$

Substituting Equations (6. 34) into Equation (6. 33) yields the expansion of $R_{n'}(t, s)$:

$$R_{n'}(t, s) = \sum_{i, j, k, l, m, p}^{N} a_{ij} a_{kl}^* r_{mp} \int_0^T \int_0^T \phi_i(t) \psi_k^*(s)\, \phi_l^*(u)\, \phi_m(u)\, \psi_j(\tau)\, \psi_p^*(\tau)\, du\, d\tau$$

$$= \sum_{i, j, k, l} a_{ij}\, a_{kl}^*\, r_{lj}\, \phi_i(t)\, \psi_k^*(s)$$

$$= \sum_{i, k} b_{ik}\, \phi_i(t) \psi_k^*(s) \quad , \tag{6. 35}$$

where

$$b_{ik} = \sum_{j,l} a_{ij}\, a^{*}_{kl}\, r_{lj} \qquad . \qquad (6.36)$$

Since $\delta(t-s)$ is not in L^2, the expansion of Equation (6.35) can be an expansion of a high thin pulse which is in L^2 and is an approximation to $\delta(t-s)$. The coefficients of $R_{n'}(t, s)$ are approximately

$$b_{ik} = \begin{cases} 0 \text{ for } i \neq k , \\ 1 \text{ for } i = k \end{cases} \qquad . \qquad (6.37)$$

The N^2 unknowns a_{ij} of the N^2 equations (6.37) are the coefficients of the time-varying whitening filter.

For real $n(t)$, $R_n(\tau, u)$ is symmetric. Therefore a standard eigenfunction approach can be used to solve the integral equation (6.33). Let $\{\phi_j\}$ be the set of eigenfunctions of $R_n(\tau, u)$,

$$\lambda_j\, \phi_j(\tau) = \int_0^T R_n(\tau, u)\, \phi_j(u)\, du \qquad . \qquad (6.38)$$

Since the set of eigenfunctions is complete, the system functions can be expanded as

$$k(t, \tau) = \sum_j a_j(t)\, \phi_j(\tau) \qquad , \qquad (6.39)$$

where

$$a_j(t) = \int_0^T k(t, \tau)\, \phi_j(\tau)\, d\tau \qquad . \qquad (6.40)$$

By using the expansion of Equation (6. 39), Equation (6. 33) becomes:

$$R_{n'}(t, s) = \sum_j a_j(t) \int_0^T \int_0^T k^*(s, u)\, R_n(\tau, u)\, \phi_j(\tau)\, d\tau\, du$$

$$= \sum_j \lambda_j\, a_j(t) \int_0^T k^*(s, u)\, \phi_j(u)\, du$$

$$= \sum_j \lambda_j\, a_j(t)\, a_j^*(s) \qquad . \qquad (6.41)$$

If one first considers making the output noise spectrum to be only stationary:

$$R_{n'}(t, s) = R(t - s) \qquad ; \qquad (6.42)$$

then $a_j(t)$ is identified as $(\gamma_j/\lambda_j)^{1/2}\, \psi_j(t)$, where $\psi_j(t)$ and γ_j are the eigenfunction and eigenvalue associated with $R(t - s)$:

$$\gamma_j\, \psi_j(t) = \int_0^T R(t - s)\, \psi_j(s)\, ds \qquad . \qquad (6.43)$$

The filter given by the expansion of Equation (6. 39) makes the input noise stationary; a time-invariant whitening filter in cascade with $k(t, \tau)$ makes the noise white.

One physical model of this process of whitening follows. White stationary noise passes through some time-varying filter $k^{-1}(t, \tau)$ to become nonstationary. The whitening filter is essentially the right-hand inverse to $k^{-1}(t, \tau)$. Finding $k(t, \tau)$ from $R_n(t, \tau)$ is the same as identifying $k^{-1}(t, \tau)$ from $R_n(t, \tau)$ and then constructing its right-hand inverse $k(t, \tau)$.

F. MATCHED FILTER TO A SIGNAL IN NONSTATIONARY NOISE

If the whitening filter $k(t, \tau)$ has a left-hand inverse $k^{-1}(t, \tau)$, the nonstationary noise $n(t)$ can be considered as having been generated by passing white stationary noise through $k^{-1}(t, \tau)$. By using $k^{-1}(t, \tau)$ as a pretransmission filter, the matched filter for a channel with nonstationary noise is constructed as shown in Figure 6.9. This matched filter has components at both the transmitting and receiving ends. It is desirable, however, not to have a pretransmission filter because one would like to transmit a standard signal and not worry about time-varying average and peak power restrictions.

A reasonable detection scheme has the whitening filter followed by a matched filter to the signal component of the output of the whitening filter, which, in turn, is a matched filter to $y_{\tau_o}(t)$, where

$$y_{\tau_o}(\tau) = \int_{\tau_o}^{t} k(t, \tau)\ x(\tau)\ d\tau \qquad (6.44)$$

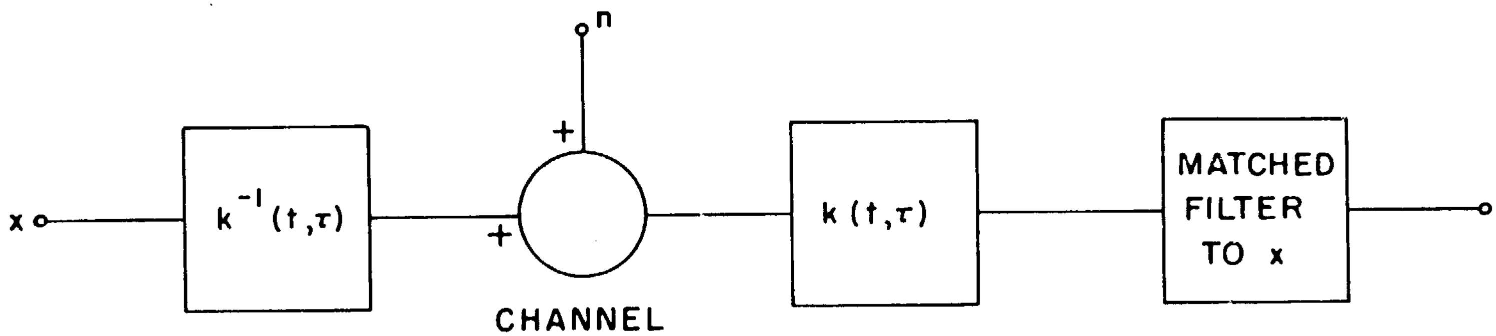


FIGURE 6.9. Matched Filter No. 1 for Channel with Nonstationary Noise.

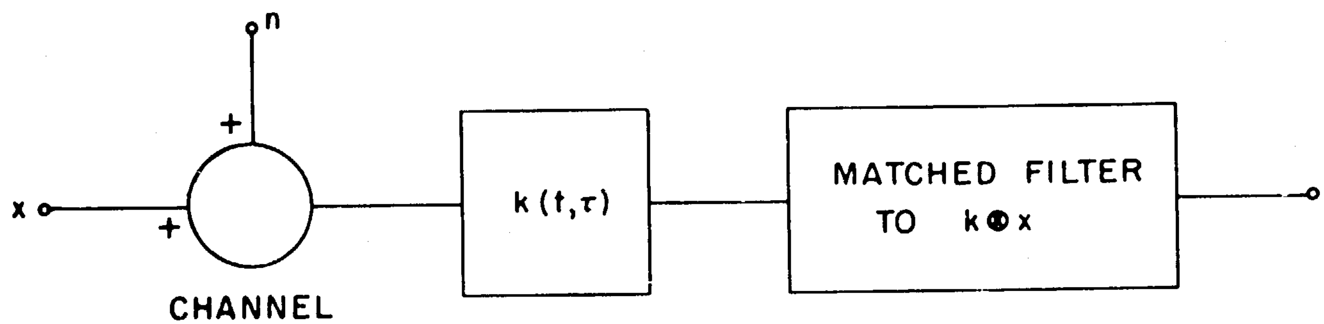


FIGURE 6.10. Matched Filter No. 2 for Channel with Nonstationary Noise.

Since $y_{\tau_o}(t)$ depends on the initial time of transmission τ_o, the matched filter itself is time-varying. In the case where transmission will occur only at discrete times τ_i, a bank of matched filters, each matched to $y_{\tau_i}(t)$, can be used. Now one can show that this detection scheme, as given in Figure 6.10, is indeed the matched filter for a channel with nonstationary noise.

A filter $f(t, \tau)$ is a matched filter to a signal $x(t)$ if the ratio of the instantaneous output power of the signal component at time t_o to the average output noise power at time t_o be a maximum. The instantaneous output signal power at time t_o is:

$$P_{y'_{\tau_o}}(t_o) = \left[y'_{\tau_o}(t_o)\right]^2 = \left[\int_{\tau_o}^{t_o} f(t_o, \tau)\, x(\tau)\, d\tau\right]^2 . \tag{6.45}$$

The average output noise power at time t_o is

$$E\left[n'(t_o)^2\right] = R_{n'}(t_o, t_o) = N(t_o) . \tag{6.46}$$

Also, the noise component of the output of $f(t, \tau)$ at time t is

$$n'(t) = \int_{-\infty}^{t} f(t, \tau)\, n(\tau)\, d\tau , \tag{6.47}$$

and the autocorrelation of the output noise is

$$R_{n'}(t, u) = \int_{-\infty}^{t} \int_{-\infty}^{u} f(t, \tau)\, f^*(u, r)\, R_n(\tau, r)\, d\tau\, dr . \tag{6.48}$$

Under the assumption that n(t) is generated by passing white stationary noise $n''(t)$ through a filter $k^{-1}(t, \tau)$, i.e.

$$n(t) = \int_{-\infty}^{t} k^{-1}(t, \tau)\, n''(\tau)\, d\tau \qquad , \tag{6.49}$$

$R_n(\tau, r)$ becomes

$$R_n(\tau, r) = \int_{-\infty}^{\tau} \int_{-\infty}^{r} k^{-1}(\tau, u)\, k^{*-1}(r, v)\, R_{n''}(u, v)\, du\, dv \qquad . \tag{6.50}$$

Since $n''(t)$ is white,

$$R_{n''}(u, v) = \delta(u-v) \qquad , \tag{6.51}$$

and $R_n(\tau, r)$ becomes

$$R_n(\tau, r) = \int_{-\infty}^{\tau} k^{-1}(\tau, v)\, k^{*-1}(r, v)\, dv \qquad . \tag{6.52}$$

The average noise power at time t_o then becomes

$$R_{n'}(t_o, t_o) = \int_{-\infty}^{t_o} \int_{-\infty}^{t_o} \int_{-\infty}^{t_o} f(t_o, \tau)\, f^*(u, r)\, k^{-1}(\tau, v)\, k^{*-1}(r, v)\, d\tau\, dr\, dv$$

$$= \int_{-\infty}^{t_o} \left| \int_{-\infty}^{t_o} f(t_o, \tau)\, k^{-1}(\tau, v)\, d\tau \right|^2 dv \qquad . \tag{6.53}$$

The ratio to be maximized is

$$\frac{P_y(t_o)}{N(t_o)} = \frac{\left[\int_{\tau_o}^{t_o} f(t_o, \tau)\, x(\tau)\, d\tau\right]^2}{\int_{-\infty}^{t_o} \left| \int_{-\infty}^{t_o} f(t_o, \tau)\, k^{-1}(\tau, v)\, d\tau \right|^2 dv} \quad , \tag{6.54}$$

which can be simplified by making the change of variables:

$$f(t, \tau) = g(t) \otimes k(t, \tau) \quad . \tag{6.55}$$

Thus,

$$\begin{aligned} \int_{-\infty}^{t} f(t, \tau)\, k^{-1}(\tau, v) d\tau &= \Big(g(t) \otimes k(t, \tau)\Big) \otimes k^{-1}(\tau, v) \\ &= g(t) \otimes \Big(k(t, \tau) \otimes k^{-1}(\tau, v)\Big) \\ &= g(t) \otimes \delta(t-v) = g(t-v) \quad , \end{aligned} \tag{6.56}$$

and

$$N(t_o) = \int_{-\infty}^{t_o} |g(t_o - v)|^2\, dv \quad . \tag{6.57}$$

The numerator becomes

$$P_y(t_o) = \left[\int_{-\infty}^{t_o} g(t_o - v) \left(\int_{\tau_o}^{v} k(v, \tau)\, x(\tau)\, d\tau\right) dv\right]^2 \quad , \tag{6.58}$$

but by Schwartz's inequality,

$$P_y(t_o) \leq \int_{-\infty}^{t_o} \left|g(t_o-v)\right|^2 dv \int_{-\infty}^{t_o} \left| \int_{\tau_o}^{v} k(v, \tau)\, x(\tau)\, d\tau \right|^2 dv \,. \tag{6.59}$$

The maximum bound on the ratio of Equation (6.54) is therefore

$$\frac{P_y(t_o)}{N(t_o)} \leq \int_{-\infty}^{t_o} \left| \int_{\tau_o}^{v} k(v, \tau)\, x(\tau)\, d\tau \right|^2 dv \quad . \tag{6.60}$$

This maximum occurs when $g(t)$ is chosen as

$$g(t_o-v) = \int_{\tau_o}^{v} k(v, \tau)\, x(\tau)\, d\tau = y_{\tau_o}(v) \quad , \tag{6.61}$$

or

$$g(t) = y_{\tau_o}(t_o-t) \quad . \tag{6.62}$$

The impulse response $g(t)$ is the impulse response of the matched filter to $y_{\tau_o}(t)$. Considering the change of variables of Equation (6.55), the matched filter for a channel with nonstationary noise is seen to be

$$f(t, \tau) = y_{\tau_o}(t_o-t) \otimes k(t, \tau) \quad . \tag{6.63}$$

Equation (6.63) represents the whitening filter cascaded with the matched filter to the signal component of the whitening filter output.

The maximum bound given by Equation (6.60),

$$M(t_o, \tau_o) = \int_{-\infty}^{t_o} \left| y_{\tau_o}(v) \right|^2 dv \quad , \qquad (6.64)$$

is a function of both t_o and τ_o. From the form of $M(t_o, \tau_o)$ one sees that $M(t_o, \tau_o)$ increases with increasing t_o in the same way as in the time-invariant case. For fairly large t_o, $M(t_o, \tau_o)$ is essentially the energy in $y_{\tau_o}(t)$. This suggests that optimization of the signal transmission time, for a fixed signal $x(t)$, be done so that the signal is transmitted at a time τ_o for which the output of the whitening filter $y_{\tau_o}(t)$ has maximum energy. This corresponds to choosing τ_o at a time when the noise power is low, in some sense. The optimization of τ_o should be done in conjunction with optimization of the shape of the signal $x(t)$ for a fixed transmission time τ_o or for a given <u>a priori</u> distribution of cost function over τ_o.

1. Optimum Signal Design for Slowly Varying Noise

For the case where the duration of the signal $x(t)$ is short enough so that the noise has fairly constant statistics over the duration of the signal, the noise can be approximated in a stepwise manner. The noise can be considered as stationary on the time intervals $\{\Delta_k\}$ with corresponding spectral density $\{N_k(\omega)\}$, where $\Sigma\, \Delta_k$ is the time interval of interest. Then the time-varying whitening filter on the time interval Δ_k becomes simply the time invariant whitening filter $W_k(\omega)$,

where

$$\left|W_k(\omega)\right|^2 = \frac{1}{N_k(\omega)} \qquad . \tag{6.65}$$

The optimum signal maximizes the quantity $M(t_o, \tau_o)$. For large t_o and $\tau_o = \tau_k \in \Delta_k$, $M(t_o, \tau_k)$ is approximately

$$M(\infty, \tau_k) = \int_{-\infty}^{\infty} \left|X(\omega)\; W_k(\omega)\right|^2 \, d\omega \qquad . \tag{6.66}$$

$M(\infty, \tau_k)$ is maximized by taking

$$X(\omega) = a\; W_k(\omega)\; e^{j\omega\tau_k} \qquad , \tag{6.67}$$

where a is a constant, and then by choosing the index k_o for which

$$M(\infty, \tau_k) = \int_{-\infty}^{\infty} \left|W_k(\omega)\right|^4 \, d\omega \tag{6.68}$$

is a maximum. Equation (6.67) gives the optimum signal waveshape, and Δ_{k_o} is the optimum time interval during which the signal should be transmitted. The value of k_o for which Equation (6.68) is a maximum corresponds to the time interval Δ_{k_o} when the noise power is lowest.

It should be mentioned in conjunction with signal optimization that there is essentially no difference between the first and second matched filters, as shown in Figures 6.9 and 6.10. If $x_o(t-\tau)$ is the optimum signal for the second matched filter, then

$$x_o'(t-\tau) = \int_{\tau}^{t-\tau} k(t-\tau, u)\; x_o(u-\tau)\, du \qquad (6.69)$$

is the corresponding optimum signal for the first matched filter.

If the signal for the second filter $x_1(t-\tau)$ is not optimum and if $M(\infty, \tau)$ is greater for the first matched filter system, when $\alpha x_1(t-\tau)$ is its input (where the constant α is chosen to that the transmitter of either matched filter has identical average power), then it is obvious that the output of the pretransmission filter

$$x_1' (t-\tau) = \int_{\tau}^{t-\tau} k^{-1}(t-\tau, u)\; x_1(u-\tau)\, du \qquad (6.70)$$

is a better signal to use for the second matched filter system.

G. CONCLUSIONS

In this investigation a number of methods were developed for analyzing and synthesizing time-varying systems. The characterization of time-varying systems was completed by the definition of the K system functions and the complimentary system functions. The relationships among the system functions were clarified, and the physical interpretation of the K system functions as impulse responses was seen to be useful in finding the system function for a cascaded system. The time-frequency duality relationships among the system functions were noted, and the introduction of physical variables enlarged the concept of duality

so that knowing one relationship becomes equivalent to knowing four relationships instead of two.

The terminated multiplier structure was shown to be separable for an exponential multiplier or for a periodic multiplier when one of the terminating networks has a periodic frequency response. The analysis of the terminated multiplier was applied to a modulation-demodulation system.

The system functions were expanded in a double series in terms of the complete set $\left\{\phi_i(x)\psi_j(y)\right\}$, as in a double sampling series. Networks were presented to evaluate the coefficients of these expansions. By using the system function expansion, the input-output equation was shown to be reducible to a matrix relationship.

The conditions for the invertibility of a time-varying delay system were found, and the general inversion problem was discussed from the matrix point of view. The known theorems on the existence of inverses and quasi-inverses were presented and were extended for the recoverability of a signal of finite duration. The application of the double series to the integral eigenvector equation reduced it to a matrix eigenvector equation. It was suggested that a judicious choice of expansion functions could lead to faster convergence than Fredholm's method.

Based on the double expansion, a method was presented for synthesizing system function by a parallel combination of separable networks where one has some control over both the networks and the multiplier. By considering the double series as a bilinear form, a unique realization was found in which a resistive cross-coupling network connected the

networks to the multipliers so that only N networks and N multipliers were used to achieve an N^2 term expansion. A synthesis scheme using the link structure and based on the sampling expansion was also presented.

The system function for a satellite communication was found, and correction for unwanted Doppler shift and time delay in an air-to-ground communication system and in a side-looking radar system were suggested. The problem of finding a whitening filter for nonstationary noise was formulated. The matched filter for a channel with nonstationary noise was shown to consist of a whitening filter followed by a matched filter to the signal component of the output of the whitening filter. Because the whitening filter is time-varying, the signal component of its output is different for different input signal starting times and thus the matched filter varies with time. When the noise varies slowly enough to appear constant over the time duration of the signal, the optimum signal is the impulse response, delayed by τ, of a constant whitening filter. The transmission time τ is chosen at a time when the noise power is low.

APPENDIX. SAMPLING THEOREMS

A. SAMPLING THEOREM IN ONE DIMENSION

Let f(t) have a Fourier transform $F(\omega)$ where $F(\omega)$ is zero outside the band $|\omega| \leq 2\pi B$. In this band, $F(\omega)$ can be expanded in a Fourier series:

$$F(\omega) = \begin{cases} \sum_{-\infty}^{\infty} a_k e^{-\frac{ikw}{2B}} & \text{for } |\omega| \leq 2\pi B , \\ 0 & \text{for } |\omega| > 2\pi B , \end{cases} \tag{A. 1}$$

where

$$a_k = \frac{1}{4\pi B} \int_{-2\pi B}^{2\pi B} F(\omega)\, e^{\frac{ikw}{2B}}\, dw$$

$$= \frac{1}{2B}\, f\left(\frac{k}{2B}\right) . \tag{A. 2}$$

Thus,

$$F(\omega) = \begin{cases} \frac{1}{2B} \sum_{-\infty}^{\infty} f\left(\frac{k}{2B}\right) e^{-\frac{ikw}{2B}} & \text{for } |\omega| \leq 2\pi B , \\ 0 & \text{for } |\omega| > 2\pi B , \end{cases} \tag{A. 3}$$

and

$$f(t) = \frac{1}{2B} \sum_{-\infty}^{\infty} f\left(\frac{k}{2B}\right)\left\{\frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} e^{iw\left(t - \frac{k}{2B}\right)} dw\right\}$$

$$= \sum_{-\infty}^{\infty} f \; \frac{k}{2B} \; \frac{\sin 2\pi B\left(t - \frac{k}{2B}\right)}{2\pi B\left(t - \frac{k}{2B}\right)} \quad . \tag{A. 4}$$

This last equation was viewed by Middleton[6] as an interpolation formula with the weighting function $(\sin 2\pi Bt)/2\pi Bt$. He postulated the following general interpolation formula in which $f(t)$ and $g(t)$ are assumed to be Fourier transformable:

$$f(t) = \sum_{-\infty}^{\infty} f(t_k)\, g(t - t_k) \quad . \tag{A. 5}$$

Equation (A. 3) can be rewritten as

$$f(t) = \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau) \quad \delta(\tau - t_k)\, d\tau \quad . \tag{A. 6}$$

and by reversing the order of summation and integration,

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t - \tau) \sum_{-\infty}^{\infty} \delta(\tau - t_k)\, d\tau \quad . \tag{A. 7}$$

The impulse train can be expanded in the Fourier series:

$$\sum_{-\infty}^{\infty} \delta(t - t_k) = \frac{1}{T} \sum_{-\infty}^{\infty} e^{-\frac{2\pi ikt}{T}} \quad , \tag{A. 8}$$

where $t_k = kT$. Now $f(t)$ is seen to be a sum of convolutions of $g(t)/T$ and $f(t)\,e^{-2\pi ikt/T}$, therefore the Fourier transform of $f(t)$ becomes

$$F(\omega) = \frac{G(\omega)}{T} \sum_{-\infty}^{\infty} F\left(\omega + \frac{2\pi k}{T}\right) \quad . \tag{A.9}$$

If one chooses $G(\omega)$ in such a way that Equation (A. 9) is an identity, then Equation (A. 5) will also be an identity and $g(t)$ will be a suitable interpolation function. Equation (A. 9) is an identity for any $F(\omega)$ limited to the band $|\omega| \leq 2\pi B$ if

$$T \leq \frac{1}{2B} \quad , \tag{A.10}$$

and

$$G(\omega) = \begin{cases} T & \text{for } |\omega| \leq 2\pi B \quad , \\ 0 \text{ for } \frac{2\pi k}{T} - 2\pi B \leq |\omega| \leq \frac{2\pi k}{T} + 2\pi B \, , & \text{for } k = 1, 2, \ldots \end{cases} \quad . \tag{A.11}$$

If $T = 1/2B$ and $F(\omega) = 0$ at only a countable number of points in the band $|\omega| \leq 2\pi B$, then the interpolation function $g(t)$ is unique because

$$G(\omega) = \begin{cases} \frac{1}{2B} & \text{for } |\omega| \leq 2\pi B \quad , \\ 0 & \text{for } |\omega| > 2\pi B \quad , \end{cases} \tag{A.12}$$

except at a countable number of points, and thus

$$g(t) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} \left(\frac{1}{2B}\right) e^{i\omega t}\, d\omega = \frac{\sin 2\pi Bt}{2\pi Bt} \quad . \tag{A.13}$$

Note that if the function is sampled at faster than $1/2B$ seconds, on the intervals

$$\frac{2\pi k}{T} + 2\pi B < \omega < \frac{2\pi(k+1)}{T} - 2\pi B \quad \text{for } k = 0, 1, 2, \ldots \quad ,$$

$$F\left(\omega + \frac{2\pi k}{T}\right) = 0 \quad . \tag{A.14}$$

From Equation (A.9) it is seen easily that $G(\omega)$ is arbitrary on these intervals. For a particular $F(\omega)$, these intervals are part of the total set

$$\left\{ \omega \;\middle|\; \sum_{-\infty}^{\infty} F\left(\omega + \frac{2\pi k}{T}\right) = 0 \;, \quad T < \frac{1}{2B} \right\}$$

on which $G(\omega)$ is arbitrary. For example, let

$$G(\omega) = \begin{cases} T & \text{for } |\omega| \leq 2\pi B \quad , \\ a & \text{for } 2\pi B < |\omega| < \frac{2\pi}{T} - 2\pi B \quad , \\ 0 & \text{otherwise} \quad , \end{cases} \tag{A.15}$$

in which case

$$g(t) = (T - a) \frac{\sin 2\pi Bt}{\pi t} + a \frac{\sin \left(\frac{2\pi}{T} - 2\pi B\right) t}{\pi t} \quad . \tag{A.16}$$

Whereas the interpolation function of Equation (A. 13) when shifted by t_k is orthogonal and $g(t - t_k)$ is zero at the other sampling times $t_j (j \neq 0)$, and thus a truncated expansion must agree with $f(t)$ at the sampling time t_k, the function of Equation (A. 16) when shifted by t_k is not orthogonal, and $g(t - t_k)$ may be nonzero at other sampling times $t_j (j \neq k)$. Therefore a truncated expansion would not necessarily agree with $f(t)$ at the sampling times t_k.

Petersen and Middleton[5] used the arbitrariness of the function $g(t)$ as an essential feature of their extension of the sampling theorem to Euclidean N-space.

B. THE SAMPLING THEOREM IN EUCLIDEAN N-SPACE

Consider the function $f(\vec{x}) = f(x_1, x_2, \ldots, x_n)$ whose Fourier transform $F(\vec{\omega})$ exists as

$$F(\vec{\omega}) = F(\omega_1, \omega_2, \ldots, \omega_n) = \int_X f(\vec{x})\, e^{-i\vec{\omega}\cdot\vec{x}}\, d\vec{x} \quad . \tag{A. 17}$$

$F(\vec{\omega})$ is said to be "wave-number limited" if it vanishes outside of a finite subspace R of "wave-number space" Ω. In this case one can define a basis

$$\left\{\vec{v}_j\right\} = \left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\} \quad , \tag{A. 18}$$

in terms of which the lattice sampling points are expressable as

$$\vec{v}_{[l]} = l_1\vec{v}_1 + l_2\vec{v}_2 + \ldots + l_n\vec{v}_n \quad ,$$

$$l_1, l_2, \ldots, l_n = 0 \pm 1, \pm 2, \ldots \quad . \tag{A.19}$$

It is desirable to find the conditions on $g(t)$ which make it an interpolation function such that $f(\vec{x})$ is expandable in the series

$$f(\vec{x}) = \sum_{[l]} f(\vec{y}_{[l]})\, g(\vec{x} - \vec{v}_{[l]}) \quad . \tag{A.20}$$

By using the N-dimensional Dirac delta function, Equation (3.22) can be written:

$$f(\vec{x}) = \int_X f(\vec{\rho})\, g(\vec{x} - \vec{\rho}) \sum_{[l]} \delta(\vec{\rho} - \vec{v}_{[l]})\, d\vec{\rho} \quad . \tag{A.21}$$

It can be shown that

$$\sum_{[l]} \delta(\vec{\rho} - \vec{y}_{[l]}) = \sum_{[m]} \frac{1}{Q} e^{-i\vec{\rho}_o \vec{u}_{[m]}} \quad , \tag{A.22}$$

where Q is the hypervolume of a parallelepiped with edges $\vec{v}_j$ and $\vec{u}_k$ related by

$$\vec{v}_j \cdot \vec{u}_k = 2\pi \delta_{jk} \quad , \tag{A.23}$$

where δ_{jk} is the Kronecker delta. By using Equation (A.22), Equation (A.21) becomes

$$f(\vec{x}) = \sum_{[m]} \int_X f(\vec{\rho})\, e^{i\vec{\rho} \cdot \vec{u}_{[m]}} \, \frac{g(\vec{x} - \vec{\rho})}{Q}\, d\vec{\rho} \quad . \tag{A.24}$$

As in the one-dimensional case, Equation (A. 24) is recognized as a sum of convolutions. Taking the Fourier transform of both sides yields

$$F(\vec{\omega}) = \frac{G(\vec{\omega})}{Q} \sum_{m} F(\vec{\omega} + \vec{u}_{[m]}) \quad . \tag{A. 25}$$

Conditions on $G(\vec{\omega})$ and $\vec{u}_k$ for which Equation (A. 25), and subsequently Equation (A. 20), are identities follow:

1. The vectors $\vec{u}_k$ must be large enough and so oriented that the shifted spectra do not overlap.

2. Let

$$G(\omega) = \begin{cases} Q \text{ for } \vec{\omega} \in R \ , \\ \\ 0 \text{ for } \omega \in R_{[m]} \ , \end{cases} \tag{A. 26}$$

where $R_{[m]}$ is the subset R shifted by the vector $\vec{u}_{[m]}$:

$$R_{[m]} = \left\{ \vec{\omega} \middle| \vec{\omega} - \vec{u}_{[m]} \in R \right\} \quad . \tag{A. 27}$$

$R_{[m]}$ is the subset on which $F(\vec{\omega} + \vec{u}_{[m]})$ may be nonzero. For any $F(\omega)$ which vanishes outside R, $G(\vec{\omega})$ is arbitrary on the compliment of the set

$$\Omega \left(\bigcup_{[n]} R_{[m]} \right) \quad .$$

$G(\vec{\omega})$ will be unique if

$$\bigcup_{[m]} R_{[m]} = \Omega \quad , \tag{A. 28}$$

but because of the shape of R, one may not be able to choose the vectors $\left\{\vec{u}_k\right\}$ such that Equation (A. 28) is satisfied. If not, $G(\vec{\omega})$ will be arbitrary over some nonzero subset of Ω. In the one-dimensional case, however, one could always choose $T = 1/2B$ (T is the length of the vector u_1) in such a way that the domains of $F(\omega + 2\pi k/T)$ covered the set $\Omega = (-\infty, \infty)$ and $G(\omega)$ was unique.

The N-dimensional sampling theorem can now be stated: A function $f(\vec{x})$ whose Fourier transform $F(\vec{\omega})$ vanishes outside of a finite subset R of wave-number space Ω can be reproduced from its sample values taken over a lattice of points $\left\{l_1 v_1 + l_2 v_2 + \dots + l_n v_n\right\}$ for $l_1, l_2, \dots, l_n = 0, \pm 1, \pm 2, \dots$ where the vectors $\left\{\vec{v}_j\right\}$ are chosen to assure nonoverlapping of the subsets $R_{[m]}$ of Ω, where $R_{[m]}$ is the set R shifted by the vector $\vec{u}_m$ and $\vec{v}_j \cdot \vec{u}_k = 2\pi\delta_{jk}$.

In the case where R is a parallelepiped centered at the origin, a particularly simple interpolation function results:

$$g(x) = \prod_{k=1}^{N} \frac{\sin \frac{1}{2} \vec{u}_k \cdot \vec{x}}{\frac{1}{2} \vec{u}_k \cdot x} \quad . \tag{A. 29}$$

If R is rectangular and centered at the origin and u_k is chosen in the direction of the k^{th} unit vector, then

$$g(x) = \prod_{k=1}^{N} \frac{\sin \frac{1}{2} u_k x_k}{\frac{1}{2} u_k x_k} \quad . \tag{A. 30}$$

Equation (A. 30) is the interpolation function that can be considered the simplest extension from one to N dimensions. Actually this interpolation

function can be used to reconstruct any function $f(\vec{x})$ whose Fourier transform $F(\vec{\omega})$ vanishes outside of an irregularly shaped finite region R' by choosing a rectangular region R centered at the origin which is large enough to include R'. The conditions of the theorem are satisfied for the larger region R.

Although the simplicity of the sampling function of Equation (A. 30) makes it especially useful, it should be mentioned that choosing those lattice vectors may lead to a relatively inefficient sampling lattice. An effective sampling lattice is one in which the vectors $\left\langle \vec{u}_j \right\rangle$ are chosen to "pack" the spectra $\left\langle F(\vec{\omega} + \vec{u}_{[m]}) \right\rangle$ as close as possible to each other without overlapping, thus assuring that the least number of samples per unit hypervolume in X space are needed. For example, in one dimension for $T = 1/2B$ the spectra $F(\omega + 2\pi k/T)$ are adjacent. This sampling time is the slowest possible; thus the least number of samples per unit time are required.

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